NON-REALIZABILITY OF BRAID GROUPS BY DIFFEOMORPHISMS

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ABSTRACT

NON-REALIZABILITY OF BRAID GROUPS BY DIFFEOMORPHISMS

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The mapping class group $\mathrm{Mod}(\Sigma_g)$ is the group of isotopy classes of orientation preserving diffeomorphisms of Σ_g. The realization problem asks if a given subgroup $\Gamma \hookrightarrow \mathrm{Mod}(\Sigma_g, \mathbf{z})$ lifts to $\mathrm{Diff}^+(\Sigma_g, \mathbf{z})$ where Σ_g is a closed orientable surface, and $\mathrm{Mod}(\Sigma_g, \mathbf{z})$ is the mapping class group of Σ_g with n marked points. Morita's non-lifting theorem gives a negative answer to the realization problem for infinite subgroups of the mapping class group $\mathrm{Mod}(\Sigma_g)$. In this thesis, we focus on two different proofs of this theorem one due to Bestvina, Church and Souto [21], and the other due to Salter and Tshishiku [22]. For this purpose, in the first proof, we will consider the case that Γ is an arbitrary finite index subgroup of the mapping class group directly [21], and we will consider the result with different numbers of marked points of Σ_g as well. In the second proof, we will consider the case that Γ is a braid group, i.e., $\Gamma = B_n$ [22], then use this to prove Morita's result.

Keywords: Mapping Class Groups, Braid Groups, Orientation Preserving Diffeomorphisms, Morita's Non-lifting Theorem

ÖZ

ÖRGÜ GRUPLARININ DİFEOMORFİZMALARLA GÖSTERİLMEMESİ

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Gönderim sınıf grubu $\mathrm{Mod}(\Sigma_g)$, Σ_g'nin yön koruyan difeomorfizmalarının izotopi sınıflarının grubudur. Gösterim problemi, verilen bir $\Gamma \hookrightarrow \mathrm{Mod}(\Sigma_g, \mathbf{z})$ alt grubunun $\mathrm{Diff}^+(\Sigma_g, \mathbf{z})$'a yükselip yükselmediğini sorar, burada Σ_g kapalı, yönledirilebilir bir yüzey ve $\mathrm{Mod}(\Sigma_g, \mathbf{z})$, Σ_g'nin n işaretli noktalı gönderim sınıf grubudur. Morita'nın yükseltilmeme teoremi, gönderim sınıf gruplarının sonsuz altgurpları için gösterim problemine olumsuz bir cevap verir. Bu tezde, bu teoremin biri Bestvina, Church ve Souto [21]'ya ve diğeri Salter ve Tshishiku [22]'ya ait olan iki farklı kanıtına odaklanıyoruz. Bu amaçla, ilk kanıtta direkt olarak Γ'nın gönderim sınıf grubunun rastgele bir sonlu indeksli alt grubu olduğu durumu ele alacağız [21] ve ayrıca farklı sayıda işaretli nokta ile elde edilen sonucu inceleyeceğiz. İkinci kanıtta, Γ'nın bir örgü grubu olduğu durumu ele alacağız, yani $\Gamma = B_n$ [22], ardından bunu Morita'nın sonucunu kanıtlamak için kullanacağız.

Anahtar Kelimeler: Gönderim Sınıf Grupları, Örgü Grupları, Oryantasyon Koruyan Difeomorfizmalar, Morita'nın Yükseltilmeme Teoremi

To deity

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$f(X)$	The image of the set X under the function f
$A{:=}B$	A is defined as B
$\mathrm{Fix}(f)$	The set of points that are fixed under the function f
$\mathbb{R}^+$	Positive real numbers
$P_+T_{\widetilde{z}}\mathbb{H}^2$	The space of directions $P_+T_{\widetilde{z}}\mathbb{H}^2{:=}(T_{\widetilde{z}}\mathbb{H}^2 \setminus \{0\})/\mathbb{R}^+$ of $\mathbb{H}^2$ at $\widetilde{z}$

0.1 Algebra

$[H:G]$	The index of the subgroup H in the group G
$A \cong B$	The group A is isomorphic to the group B
S_n	The symmetric group of degree n
A_n	The alternating group of degree n
$H \leq G$	The group H is a subgroup of the group G
$H \trianglelefteq G$	The group H is a normal subgroup of the group G
$\langle a, b, c, \ldots n \rangle$	The group generated by the set of elements $\{a, b, c, \ldots, n\}$
$\mathrm{GL}_n(\mathbb{R})$	General linear group of degree n
$\mathrm{GL}_n^+(\mathbb{R})$	General linear group of degree n whose elements have positive determinants
$\mathrm{SL}_n(\mathbb{R})$	Special linear group of degree n
$\mathrm{PSL}_n(\mathbb{R})$	Projective linear group of degree n
C_G	Center of the group G
$C(G)$	Centralizer of the group G

0.2 Topology

$\mathbb{D}^2$	$2-$dimensional closed disk

Σ_g	Surface of genus g
$\pi_1(X)$	Fundamental group of X
$\mathcal{G}(\Sigma, z)$	$\{f : \Sigma \to \Sigma \mid f$ is an orientation preserving homeomorphism such that $f(z) = z,$ and f, f^{-1} are differentiable at z $\}$
$\mathrm{Diff}^+(\Sigma, \mathbf{z})$	Set of orientation preserving diffeomorphisms from the surface Σ to itself which fix $\mathbf{z}$ setwise
$\mathrm{Homeo}^+(\Sigma, \mathbf{z})$	Set of orientation preserving homeomorphisms of the surface Σ which fix the set $\mathbf{z}$ setwise
$\mathrm{Diff}^+(\Sigma, \partial\Sigma, \mathbf{z})$	The group of C^1 diffeomorphisms f from Σ to Σ preserving $\mathbf{z}$ setwise such that the restriction of f to $\partial\Sigma$ is the identity
$\mathrm{Diff}^0(\Sigma, \mathbf{z})$	The subgroup of Diff^+ whose elements are isotopic to identity via an isotopy which fixes $\mathbf{z}$ setwise
$C^n(M)$	Set of n times continuously differentiable functions having domain M
$C^\infty(M)$	Set of infinitely many differentiable functions having domain M
S^1	The unit circle
S^2	The $2-$dimensional sphere
$\Omega^1(M)$	The set of differential $1-$forms on a manifold M
$\mathrm{cell}_n(X)$	the number of $n-$dimensional cells of the CW-complex X
$e(X)$	The Euler characteristic or Euler number $e(X)$ of X
$\mathrm{int}(X)$	The interior of X
∂X	The boundary of X
$\overline{X}$	The closure of X
$H_n(X)$	n^{th} homology group of the topological space X

0.3 Hyperbolic Geometry

$\mathbb{H}^2$	The hyperbolic plane
$\partial_\infty \mathbb{H}^2$	Boundary at infinity of the hyperbolic plane
$\overline{\mathbb{H}}^2$	Closure of the hyperbolic plane

0.4 Mapping Class Groups

$\mathrm{Mod}(\Sigma)$	The mapping class group of the surface S
$\mathrm{Mod}(\Sigma, \mathbf{z})$	The subset of the mapping class group of the surface S whose elements fix the the set $\mathbf{z}$ setwise
$\mathrm{Mod}(\Sigma, z)$	The subset of the mapping class group of the surface S whose elements fix the point z
$\mathrm{PMod}(\Sigma, \mathbf{z})$	The subset of the mapping class group of the surface S whose elements fix the the set $\mathbf{z}$ pointwise
B_n	The surface braid group of n strands

CHAPTER 1

INTRODUCTION

1.1 Motivation and Problem Definition

For an orientable surface Σ_g, the mapping class group $\mathrm{Mod}(\Sigma_g)$ is the group of orientation-preserving diffeomorphisms of Σ_g up to isomorphism, and $\mathrm{Mod}(\Sigma_g, \mathbf{z})$ is the mapping class group of Σ_g with n marked points where $\mathbf{z}$ is the set of marked points. The realization problem asks if a subgroup $\Gamma \hookrightarrow \mathrm{Mod}(\Sigma_g, \mathbf{z})$ lifts to the set $\mathrm{Diff}^+(\Sigma_g, \mathbf{z})$ of orientation preserving diffeomorphisms fixing the set $\mathbf{z}$ or the set $\mathrm{Homeo}^+(\Sigma_g)$ of orientation preserving homeomorphisms, and this problem has been answered for the cases of different types of the group Γ.

Nielsen realization problem asks that for a finite subgroup Γ of $\mathrm{Mod}(\Sigma_g)$, does Γ lift to $\mathrm{Diff}^+(\Sigma_g)$ or $\mathrm{Homeo}^+(\Sigma_g)$, and it is answered affirmatively by Kerckhoff [12]. In this article, Kerckhoff showed that a finite subgroup $\Gamma \leq \mathrm{Mod}(\Sigma_g)$ does not lift to $\mathrm{Diff}^+(\Sigma_g)$ using foliations for the orientable case (i.e., the case that Σ_g is orientable), and he mentioned the non-orientable case as well. Before him, Kravetz [4] claimed to prove the same thing, but his proof was using the statement that the Teichmüller space has negative curvature. However, Masur showed that this statement was false (see [8] and [11]).

In his paper [13], Morita stated that we can ask the same question for infinite subgroups of the mapping class group as well, and showed that the answer for $\mathrm{Diff}^+(\Sigma_g)$ was negative where Σ_g is of genus $g \geq 18$. Later, in his article [16], Morita showed that $g \geq 5$ is enough to get the same answer instead of 18. Markovic showed in his paper [17] that for Σ_g a closed surface of genus $g > 5$, $\mathrm{Mod}(\Sigma_g)$ cannot be realized by $\mathrm{Homeo}^+(\Sigma_g)$. In this paper, he also stated that the result holds for $g \geq 2$ indeed.

He said that the way he used to prove the case $g > 5$ can be generalized to $g > 2$, but the case $g = 2$ needs a different technique than he used.

1.2 Contributions and Novelties

In this thesis, we try to understand two different proofs of Morita's result for $g > 5$ due to [21], [22]. The first proof uses the Euler numbers to show a finite index subgroup $\Gamma \leq \mathrm{Mod}(\Sigma_g, z)$ does not lift to the set $\mathcal{G}(\Sigma_g, z)$ of orientation preserving homeomorphisms fixing the point $z \in \Sigma_g$ which are differentiable at z with its inverse is also differentiable at z, and then generalizes this result [21]. Before introducing the second proof, we need to know the braid groups. A braid group is the group of isotopy classes of braids. This group was first defined by Emil Artin [2]. The second proof we are going to consider uses the non-realizability of braid groups by diffeomorphisms and then shows that $\mathrm{Mod}(\Sigma_g)$ does not lift to $\mathrm{Diff}^+(\Sigma_g)$ for Σ_g is a closed orientable surface of genus $g \geq 2$ [22].

1.3 The Outline of the Thesis

In Chapter 2, we give some basic definitions and properties of needed objects such as indicable groups, one-parameter subgroups, braid and mapping class groups.

Chapter 3 consists of two main sections both of which are dedicated to a particular proof of Morita's non-lifting theorem. In each section, we will see the needed theorems and prove some of them, and then get Morita's result.

In Subsection 3.1.3, we see a proof of Morita's non-lifting theorem by using Proposition 46 which says a finite index subrgoup of $\pi_1(\Sigma_g, z)$ does not lift to $\mathcal{G}(\Sigma_g, z)$ for $g \geq 2$. To prove this, we use Milnor-Wood Inequality 43, 42, and the existence of a diffeomorphism τ (in Lemma 49) and Lemma 51.

In Subsection 3.2.2, we prove the same theorem by means of braid groups and Remark 16 which is about the action of B_{n-1} on the points of B_n, non-indicability of B_n for $n \geq 5$, Thurston Stability Theorem 58 and Lemma 57 which says that every homomorphism $f : B_n \to \mathrm{GL}_n^+(\mathbb{R})$ has abelian image for $n \geq 5$. To prove this lemma, we use One Parameter Subgroups 2.1.4.1.

CHAPTER 2

PRELIMINARIES

2.1 Algebra

A group G is said to be *perfect* if it is equal to its commutator subgroup, namely $G = [G, G]$.

Let H be a subgroup of a group G.

- The index $[G : H]$ of H in G is defined as the number of left (or, equivalently right) cosets of H.
- H is said to be *of finite index* if the index of H in G is finite.

We will use the following definition to give an example in Section 2.1.1 only.

Let $X = \{x_1, \ldots, x_n\}$ be a set. Define X^{-1} as $\{x_1^{-1}, \ldots, x_n^{-1}\}$. A finite sequence of elements of $X \cup X^{-1}$ is called a *word*. If the sequence has no elements, then we call it the *empty word*. By cancelling the elements of the form $x \cdot (x^{-1})$ from any word, we get a group. Then the group G is called a *free group with generating set X*.

2.1.1 Indicable and Non-indicable Groups

Definition 1. *Let G be a group. If there is a surjective homomorphism from H to $\mathbb{Z}$ for every non-trivial finitely generated subgroup $H \leq G$, this group is called* locally indicable.

As an example of locally indicable group, consider the free group $F_{\{a,b\}}$ generated by two elements a, b. Any non-trivial subgroup of $F_{\{a,b\}}$ is a free group, and any free group can be surjected onto $\mathbb{Z}$ homomorphically (for example, one can take the homomorphism sending a generator to 1 and the other generators to 0). This means, there is a surjective homomorphism from any non-trivial finitely generated subgroup of $F_{\{a,b\}}$ to $\mathbb{Z}$ which makes $F_{\{a,b\}}$ locally indicable.

For other two examples, consider the groups $\mathbb{Q}, \mathbb{R}$ with respect to usual addition. Their non-trivial finitely generated subgroups are abelian and any element of these groups are of infinite order. Recall the *fundamental theorem of finitely generated abelian groups* which says that every finitely generated abelian group is isomorphic to a direct sum of $\mathbb{Z}^n$ and a torsion subgroup. Since the elements of an arbitrary non-trivial finitely generated subgroup H of $\mathbb{Q}$ or $\mathbb{R}$ are of infinite order, H should be isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}$, and therefore it can be surjected into $\mathbb{Z}$ homomorphically. Since H is arbitrary, this reasoning holds for any non-trivial finitely generated subgroup, and so $\mathbb{Q}, \mathbb{R}$ are locally indicable.

Definition 2. *A group G is called* strongly non-indicable *if it has a non-trivial finitely generated perfect subgroup.*

Remark 3. *Note that if a group G is strongly non-indicable, then it is locally non-indicable. Because a strongly non-indicable group G has a non-trivial finitely generated subgroup H which is perfect. Remember the fact that for an arbitrary group K and a normal subgroup $N \trianglelefteq G$, the quotient K/N is abelian if and only if $N \supseteq [K, K]$. Therefore, to have a homomorphism ϕ from H to an abelian group, we need to have $\ker(\phi) \supseteq [H, H] = H$ which gives the image of ϕ trivial. So ϕ cannot be surjective, and G cannot be locally indicable.*
If a group G has a torsion element, then it has a non-trivial finitely generated subgroup which does not surject to $\mathbb{Z}$, so that G is locally non-indicable. In particular, if G has a non-trivial finite subgroup, then it is locally non-indicable.

To get an example of a strongly non-indicable group, consider the symmetric group $S_{\mathbb{Z}}$ on $\mathbb{Z}$. The alternating group A_5 on $\{1, 2, 3, 4, 5\}$ is a non-trivial subgroup of $S_{\mathbb{Z}}$ which is finitely generated (because it is finite). Since any element of A_5 can be written as a product of commutators in A_5, it is perfect. So, $S_{\mathbb{Z}}$ is strongly non-indicable.

We will see in Proposition 54 ii) that B_n is strongly non-indicable for $n \geq 5$ (see 2.1.3).

Remark 4. *Assume G is not a locally indicable group, and H be a subgroup of G such that there is no surjective homomorphism from H to $\mathbb{Z}$. This means there is no surjection from $H/[H,H]$ to $\mathbb{Z}$. Let N be a normal subgroup of G with $H \cap N \neq H$ which means there is an element $h \in H$ with $h \notin N$. Then, $N \not\supseteq [HN, HN]$ which implies that there is no surjective homomorphism from HN/N to $\mathbb{Z}$. Since $HN/N \leq G/N$, G/N is not locally indicable as well.*
The same reasoning applies to strongly non-indicable groups also.

2.1.2 Mapping Class Groups

Dehn gave a generating set for the mapping class groups and introduced the Dehn twists. Nielsen worked on the classification of mapping class groups, later his work was improved by Thurston.

Definition 5. *For a manifold Σ, the group of isotopy classes of the orientation preserving homeomorphisms of Σ is called the* mapping class group of Σ. *This group is denoted by* $\mathrm{Mod}(\Sigma)$.

In the special case that the manifold Σ is smooth, the above definition becomes the following:
The mapping class group $\mathrm{Mod}(\Sigma)$ *is* $\pi_0(\mathrm{Homeo}^+(\Sigma))$ *which is isomorphic to* $\pi_0(\mathrm{Diff}^+(\Sigma))$ *[19].*

Definition 6. *Let Σ be a surface and $\mathbf{z} \subseteq \Sigma$ be the set of marked points of Σ (i.e., let the elements of $\mathbf{z}$ be distinguished).*

- *The set $\mathrm{Mod}(\Sigma, \mathbf{z})$ is defined as the set of mapping classes of the surface Σ which sends an element $z \in \mathbf{z}$ to an element $\widetilde{z}$ of the same set $\mathbf{z}$.*
- *The subgroup of* $\mathrm{Mod}(\Sigma)$ *consisting of the mapping classes fixing each $z \in \mathbf{z}$ is called the* pure mapping class group, *denoted by* $\mathrm{PMod}(\Sigma, \mathbf{z})$.
  We could equivalently define the pure mapping class group as

$$\mathrm{PMod}(\Sigma, \mathbf{z}) := \mathrm{Diff}^+(\Sigma, \mathbf{z}) / \mathrm{Diff}^0(\Sigma, \mathbf{z})$$

where $\mathrm{Diff}^0(\Sigma, \mathbf{z})$ *denotes the normal subgroup of* $\mathrm{Diff}^+(\Sigma, \mathbf{z})$ *whose elements are isotopic to identity via an isotopy which fixes the set* $\mathbf{z}$*.*

Theorem 7. *(Birman Exact Sequence [19]) For a surface* Σ *whose Euler characteristic* $\chi(\Sigma) < 0$ *(see Section 2.5) with a marked point* z*, the following sequence is exact:*

$$1 \to \pi_1(\Sigma, z) \to \mathrm{Mod}(\Sigma, z) = \mathrm{PMod}(\Sigma, z) \to \mathrm{Mod}(\Sigma) \to 1.$$

2.1.2.1 Some Mapping Classes

The first example of mapping classes that we will see in this subsection is a Dehn twist. There are two approaches to Dehn twists, but they are actually the same.

Definition 8. *Let* $T : S^1 \times [0,1] \to S^1 \times [0,1]$ *be defined as* $T(\theta, t) = (\theta - 2\pi t, t)$*. Let* Σ *be an oriented surface, and* $\alpha \subseteq \Sigma$ *be a simple closed curve with a regular neighborhood* N*. For an orientation preserving map* $\phi : S^1 \times [0,1] \to N$*, the homeomorphism* $T_\alpha : \Sigma \to \Sigma$ *defined as* $T_\alpha(x) = \phi \circ T \circ \phi^{-1}(x)$ *for* $x \in N$ *and* $T_\alpha(x) = x$ *for* $x \in \Sigma \setminus N$ *is called a* (right) Dehn Twist about α.

Fact 9. *The isotopy class of* T_α *does not depend on the neighborhood* N *or the isotopy class of the curve* α*. Therefore, this class is an element of* $\mathrm{Mod}(\Sigma)$*.*
We will refer to the class of T_α *as Dehn twist as well and denote it by* T_α *again.*

Let us introduce the second approach.

Definition 10. *Let* Σ *be an orientable surface, and* $\alpha \subseteq \Sigma$ *be a simple closed curve. Cut the surface* Σ *along* α*. This results in two boundary components on* Σ*. Twist a neighborhood of one of these components to the right by* 2π *radians, then glue along the curve* α*. This process gives a homeomorphism which is called a* Dehn Twist about α.

As in Fact 9, the isotopy class of α does not affect the isotopy class of T_α, and the notation will be the same as explained there.

Figure 2.1: How the Dehn twist T_α acts on a curve b

Another element of the mapping class group is the class of a hyperelliptic involution. We will need this concept for the proof of Theorem 65. Let Σ_g be a genus-g surface. Let r be a reflection of a regular $(4g+2)-$gon through its center. Then, r is a hyperelliptic involution, and the isotopy class of r is an element of $\mathrm{Mod}(\Sigma_g)$ [19].

2.1.3 Braid Groups

Although it is first defined by Artin, the concept of a braid appears in older mathematical works as well. For example, Gauss used it while studying knots [18], and Hurwitz used the concept implicitly in his paper [1] in 1891 .

Unlike Artin who defined a braid group as a collection of strands, Hurwitz considered a braid group as the fundamental group of a configuration space, and his understanding of braid groups was forgotten till it was reused by Neuwirth, Fadell and Fox in the articles [5] and [6] in 1962.

We are going to state both definitions here, but we will use the second definition. First, we shall start with the definition introduced by Artin in 1925. This is the first rigorous definition for braid groups.

Definition 11. *Let $p_1, p_2, \ldots, p_n$ be n distinct points in $\mathbb{D}^2$, and $f_i : [0,1] \to \mathbb{D}^2 \times [0,1]$ be paths such that $f_i([0,1])$ are pairwise disjoint for $i \in \{1, 2, \ldots, n\}$, and let $\sigma \in \mathrm{S}_n$ be a permutation. If*

- $f_i(t) \in \mathbb{D}^2 \times \{t\}$
- $f_i(0) = p_i \times \{0\}$
- $f_i(1) = p_{\sigma(i)} \times \{1\}$,

then the maps f_i are called strands*, and the collection $(f_1, f_2, \ldots, f_n)$ of the strands is called a* braid.

The product of two braids $(f_1(t), f_2(t), \ldots, f_n(t)), (g_1(t), g_2(t), \ldots, g_n(t))$ *is defined as*

$$(f_1(t), f_2(t), \ldots, f_n(t)) \cdot (g_1(t), g_2(t), \ldots, g_n(t)) = \begin{cases} f_i(2t) & 0 \leq t \leq \frac{1}{2} \\ g_{\sigma(i)}(2t-1) & \frac{1}{2} < t \leq 1 \end{cases},$$

see Figure 2.2.

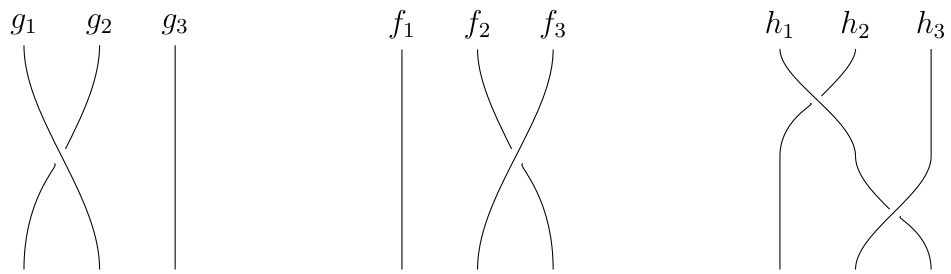

Figure 2.2: The product (h_1, h_2, h_3) of braids (f_1, f_2, f_3) and (g_1, g_2, g_3)

Finally, we can define braid groups.

Definition 12. *The group of isotopy classes of braids with the binary operation defined above is called the* braid group on n strands *which is denoted by B_n.*

Let us continue with the second definition. Here, we need to know what a configuration space is.

Definition 13. *Let X be a topological space. The set*

$$\{(x_1, x_2, \ldots, x_n) \in X^n : x_i \neq x_j \text{ whenever } i \neq j\}$$

is called the (n^{th}) configuration space *of X, and it is denoted by* $\mathrm{Conf}_n(X)$.

By means of this, we can introduce the surface braid groups.

Definition 14. *For a surface Σ, the fundamental group $B_n(\Sigma){:=}\pi_1(\mathrm{Conf}_n(\Sigma))$ is called a* surface braid group. *In the special case that $\Sigma = \mathbb{D}^2$, $B_n(\Sigma)$ is denoted by B_n.*

A specific type of surface braid group gives us the alternative definition that we were looking for.

Remark 15. *[19] Let* $\mathbf{z} = \{z_1, z_2, \ldots, z_n\}$ *be marked points of* $\mathbb{D}^2$. *Then,* $B_n \cong \mathrm{Mod}(\mathbb{D}^2, \mathbf{z})$.

Remark 16. *Let* $\mathbf{z}_i = \{z_1, z_2, \ldots, z_i\}$ *denote the set of the marked points of* $\mathbb{D}^2$.
(i) Since B_{n-1} *acts on first* $n-1$ *points (namely,* B_{n-1} *acts on the set* $\mathbf{z}_{n-1}$*), it stabilizes the* n^{th} *point in* B_n *(in other words,* B_{n-1} *fixes the set* $\mathbf{z}_n \setminus \mathbf{z}_{n-1}$*) (see Definition 14).*
(ii) By (i), B_n *stabilizes the* $(n+1)^{st}$ *point in* B_{n+1}. *The inclusion* ι *from* $(\mathbb{D}^2, \mathbf{z}_n)$ *to the sphere* S^2 *stabilizes the same point, and* ι *gives an inclusion* $B_{n+1}(\mathbb{D}^2) \hookrightarrow B_{n+1}(S^2)$. *Thus, we have an inclusion* $B_n \hookrightarrow B_{n+1}(S^2)$ *which stabilizes the* $(n+1)^{st}$ *point in* $B_{n+1}(S^2)$.

2.1.3.1 Braid Relations

The braid group B_n is generated by the elements $\sigma_i \in \pi_1(\mathrm{Conf}_n(\mathbb{D}^2))$ that change the places of the i^{th} and the $(i+1)^{\text{st}}$ points counter-clockwise, and fix the remaining points where $i \in \{1, 2, \ldots, n-1\}$ (see Figure 2.3).

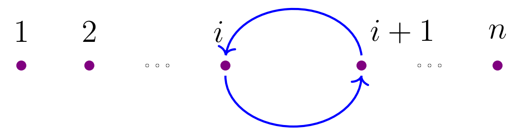

Figure 2.3: The strand generator σ_i of B_n

There are some relations between these σ_i, called the *braid relations*:

- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i \in \{1, 2, \ldots, n-2\}$
- $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$.

In fact, Artin [2] showed that the group B_n can be presented as

$$B_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} : \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } i \in \{1, 2, \ldots, n-2\}, \\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1\}\rangle.$$

2.1.4 Lie Algebra

A *Lie algebra* $\mathfrak{g}$ *over a field* F is a vector space $\mathfrak{g}$ over a field F with a binary operation, called *Lie bracket*; $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

- $[\cdot,\cdot]$ is bilinear, i.e., $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ and $[X, aY + bZ] = a[X, Y] + b[X, Z]$ for all $a, b \in F$ and for all $X, Y, Z \in \mathfrak{g}$,
- $[X, X] = 0$ for all $X \in \mathfrak{g}$, and
- the *Jacobi identity* $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \mathfrak{g}$ holds.

Two elements X and Y in the Lie algebra $\mathfrak{g}$ are said to *commute* if $[X, Y] = 0$.
The Lie algebra $\mathfrak{g}$ is called *commutative* if any two elements X and Y of $\mathfrak{g}$ commute.
A *Lie group* G is a group, which is also a finite dimensional smooth manifold such that the group operation $G \times G \to G$, $(g_1, g_2) \mapsto g_1 g_2$ and the inversion $G \to G$, $g \mapsto g^{-1}$ are smooth (i.e., infinitely many times continuously differentiable) maps.

2.1.4.1 One Parameter Subgroups

Definition 17. *Let G be a topological group (see Definition 19). A continuous group homomorphism $\phi : \mathbb{R} \to G$ is called a* one parameter subgroup.

In particular, a one parameter subgroup of $\mathrm{GL}_n(\mathbb{R})$ (or $\mathrm{PSL}_n(\mathbb{R})$) is a continuous group homomorphism $\phi : \mathbb{R} \to \mathrm{GL}_n(\mathbb{R})$ ($\phi : \mathbb{R} \to \mathrm{PSL}_n(\mathbb{R})$ respectively). We will use this concept in Chapter 3. Sometimes the image G of a one-parameter subgroup is referred as a one-parameter subgroup, and we will do so.

Remark 18. *By definition, it follows that the preimage of a one-parameter subgroup of any group is abelian.*

2.2 Topology

Definition 19. *A* topological group G *is a topological space which is also a group such that the group operation* $G \times G \to G$, $(g_1, g_2) \mapsto g_1 g_2$ *and the inversion* $G \to G$, $g \mapsto g^{-1}$ *are continuous.*

Definition 20. *Let* X *be a topological space and* $x \in X$*. A continuous map* $\gamma : [0, 1] \to X$ *satisfying* $\gamma(0) = \gamma(1) = x$ *is called a* loop based at x.

Let X *and* Y *be topological spaces, and* $f : X \to Y$ *be a continuous map.*

- *If* f *is a bijection and* f^{-1} *is continuous as well, then* f *is called a* homeomorphism.
- *The spaces* X *and* Y *are said to be* homeomorphic *if there is a homeomorphism between them.*

A subspace of a topological space is called a simple closed curve *if it is homeomorphic to the unit circle* S^1.

A closed curve which is not homotopic to a point, boundary component or puncture is called an essential curve.

Let X, Y *be two topological spaces. An* embedding $f : X \to Y$ *is a one-to-one continuous map from* X *to* Y *such that* f *gives a homeomorphism from* X *to* $f(X)$.

Definition 21. *A* $2-$*dimensional manifold is called a* surface.

Theorem 22. *(Classification of surfaces [19]) Any closed (i.e., compact without boundary), orientable surface is homeomorphic to the connected sum of a sphere and* $g \geq 0$ *many tori.*

The number g above is called the *genus* of the surface.

Definition 23. *Let* M *be a manifold.*

- *A function* $f : M \to \mathbb{R}$ *is called* differentiable at $x \in M$ *if it is differentiable around* x *in any chart.*

- *f is* differentiable *if it is differentiable at every point $x \in M$.*
- *For a bijection f between two manifolds M, N, if f and f^{-1} are continuously differentiable r times, then f is said to be a C^r*-diffeomorphism.

2.3 Algebraic Topology

Definition 24. *Let X and Y be topological spaces, and $f, g : X \to Y$ be two continuous functions.*

- *A family of maps $h_t : X \to X$ such that the associated map $H : X \times [0,1] \to Y$, $H(x,t) = h_t(x)$ is continuous and satisfies $H(x,0) = h_0 = f(x), H(x,1) = h_1 = g(x)$ for all $x \in X$ is called a* homotopy from f to g.
- *In this case, the maps f and g are said to be* homotopic *to each other.*

Remark 25. *Being homotopic is an equivalence relation. This allows us to give Definition 26.*

Let $f, g : X \to Y$ be two embeddings between two topological spaces X, Y.

- *A homotopy $h : X \times [0,1] \to Y$ with $h(x,0) = f(x)$ and $h(x,1) = g(x)$ is called an* isotopy from f to g *if $h(x,t)$ is an embedding for every $t \in [0,1]$.*
- *If there is an isotopy between the spaces X and Y, then they are said to be* isotopic *to each other.*

Definition 26. *Let X be a topological space and $\gamma_1, \gamma_2 : [0,1] \to X$ be two paths with $\gamma_1(1) = \gamma_2(0)$. The binary operation $*$ is defined as*

$$\gamma_1 * \gamma_2 = \begin{cases} \gamma_1(2t), & 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t-1), & \frac{1}{2} < t \leq 1 \end{cases}$$

is called concatenation.

Definition 27. *For the topological space X, and a point $x_0 \in X$. Concatenation operation and the set $\pi_1(X, x_0)$:=$\{[\gamma] : \gamma$ is a loop at x_0 in $X\}$, where $[\gamma]$ denotes the class of loops at x_0 in X homotopic to γ, define a group which is called the* fundamental group of X, *and the point x_0 is called the* base point.

Definition 28. *Let X and $\widetilde{X}$ be two topological spaces, and let $p : \widetilde{X} \to X$ be a continuous surjection. Suppose that for every $x \in X$, x has an open neighborhood $U \subseteq X$ such that $p^{-1}(U)$ is a union of open sets which are disjoint and their image under p is homeomorphic to U.*

Then, the space $\widetilde{X}$ with the function $p : \widetilde{X} \to X$ is called a covering space of X.

Definition 29. *Let $p : \widetilde{X} \to X$ be a covering space, and $f : \widetilde{X} \to \widetilde{X}$ be a homeomorphism. If the diagram*

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \widetilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

commutes, then f is called a deck transformation *or a* covering transformation.

Let $f : X \to Y$ be a continuous maps between topological spaces X and Y with $f(x_0) = y_0$. Then, the induced homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ of the fundamental groups is defined as $f_*([\gamma]) = [f \circ \gamma]$.

Proposition 30. *(Lifting Criterion [15]) Let Y be path-connected and locally-path connected. For a covering space $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$, a map $f : (Y, y_0) \to (X, x_0)$ lifts to $(\widetilde{X}, \widetilde{x_0})$ if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(X, x_0))$.*

Proposition 31. *(Unique Lifting Property [15]) Let $p : \widetilde{X} \to X$ be a covering space, $f : Y \to X$ be a continuous map where Y is connected, and $\widetilde{f}_1, \widetilde{f}_2 : Y \to \widetilde{X}$ be two lifts of f. If $\widetilde{f}_1(y') = \widetilde{f}_2(y')$ for some $y' \in Y$, then $\widetilde{f}_1(y) = \widetilde{f}_2(y)$ for all $y \in Y$, i.e., these two lifts are the same.*

Definition 32. *A continuous map $p : E \to B$ is said to* have homotopy lifting property with respect to the space X *if the existence of a homotopy $g_t : X \to B$ and a lift $\widetilde{g}_0 : X \to E$ of g_0 implies the existence of a lift $\widetilde{g}_t : X \to E$ of g_t.*

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\widetilde{g}_0"'] \arrow[ur, dashed, bend left, "\widetilde{g}_t"] \arrow[r, "g_t"'] & B \end{tikzcd}$$

A map $p : E \to B$ satisfying homotopy lifting property with respect to any space X is called a *fibration*.

Covering spaces (see Definition 28), projections, and fiber bundles (see Definition 33) are fibrations.

Definition 33. *Let F, B, E be topological spaces, and $p : E \to B$ be a projection map. If for all $b \in B$, there are an open neighborhood $U \subseteq B$ of b and a homomorphism $h : p^{-1}(U) \to U \times F$ which makes the diagram*

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

commute, where $\pi_1 : U \times F \to U$ is defined as $\pi_1(u, f) \mapsto u$, then p is said to be a fiber bundle*, B is called the* base space*, E is said to be the* total space*, and F is said to be a* fiber.

Let G be a topological group acting faithfully on F (i.e., for $g \in G$ and for all $f \in F$, $gf = f$ implies that g is the identity element of G). Let $U \subseteq B$ be an open set in B, and the collection of homeomorphisms $\{\phi : U \times F \to p^{-1}(U)\}$ (ϕ is said to be a chart over *U) satisfy*

- *the diagram*

$$\begin{tikzcd} U \times F \arrow[rr, "\phi"] \arrow[dr, "p_U"'] & & p^{-1}(U) \arrow[dl, "p"] \\ & U & \end{tikzcd}$$

  commutes for any chart ϕ over U,

- *for all $b \in B$, there is an open neighborhood over which there is a chart,*

- *for a chart ϕ over U and $V \subseteq U$ open, the restriction of ϕ to V is a chart over V,*

- *for any two charts ϕ and ϕ' over U, there is a continuous map $\theta_{\phi,\phi'} : U \to G$ such that $\phi'(u, f) = \phi(u, \theta_{\phi,\phi'}(u)f)$ for all $u \in U$ and for all $f \in F$,*

- *$\{\phi : U \times F \to p^{-1}(U)\}$ is maximal with respect to these properties.*

Then, G is called a structure group of F.

A principal $G-$bundle over B *is a fiber bundle $p : E \to B$ with fiber $F = G$ and structure group G acting by left translations.*

For a fiber bundle, if the fiber is the circle S^1, then it is called a circle bundle*, and if the fiber is an n dimensional plane, then it is called an $n-$*plane bundle.

A *section* of a vector bundle $p : E \to B$ is a map $s : B \to E$ such that $p \circ s = id_B$, namely the map s is a right inverse of p.

Let B be a manifold. Linear functionals on the tangent space at a point $p \in B$ are called *tangent covectors*.

The space T_p^*B of all covectors at p is called the *cotangent space* at p. The union of all cotangent spaces at all points $p \in B$ is a vector bundle called the *cotangent bundle*, denoted by T^*B.

Sections of cotangent bundle $T^*B \to B$ are called *differential* $1-$*forms on* B. The set of differential $1-$forms is denoted by $\Omega^1(B)$.

Let $p : E \to B$ be a vector bundle over a manifold B, and let $\mathcal{E}(B)$ denote the space of smooth sections of B. A *connection* in P is a map $\nabla : \Omega^1(B) \times \mathcal{E}(B) \to \mathcal{E}(B)$ such that

- $\nabla_X Y$ is linear over $C^\infty(B)$ in X, i.e., $\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y$ for $f, g \in C^\infty$,

- $\nabla_X Y$ is linear over $\mathbb{R}$ in Y, i.e., $\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2$ for $a, b \in \mathbb{R}$,

- for $f \in C^\infty(B)$, $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$

where $\nabla_X Y$ denotes the image of (X, Y) under ∇. Here, $\nabla_X Y$ is called the *covariant derivative of* Y *in the direction of* X.

For the G, p and ∇ as above, consider G as a Lie group with lie algebra $\mathfrak{g}$. The *curvature of* ∇ is defined as $F_\nabla := d\nabla + [\nabla, \nabla]$.

A vector bundle which is endowed with a connection of curvature 0 is called a *flat bundle*.

Definition 34. *Let X be a connected topological space. A simply connected topological space $\widetilde{X}$ with a covering space $p : \widetilde{X} \to X$ is called the* universal cover *of X. Let $f : \widetilde{X} \to X$ be a continuous map between two topological spaces $\widetilde{X}, X$. If there is a finite set $C \subsetneq X$ such that restriction of f to $\widetilde{X} \setminus f^{-1}(C)$ with image $X \setminus C$ is a covering, then f is said to be a* branched covering.

2.3.1 Lefschetz-Hopf Fixed Point Theorem

Let X be a topological space and $f : X \to X$ be a continuous map. The sum

$$L(f) = \sum_{i \geq 0} (-1)^i Tr(f_* : H_i(X, \mathbb{Q}) \to H_i(X, \mathbb{Q})),$$

where $Tr(f_*)$ denotes the trace of the matrix representation of the induced map f_*, is called the *Letschetz number of f*.

Let $f : X \to X$ be a continuous map, and $x_0 \in X$ be a fixed point of f. The *index* $i(f, x_0)$ *of* x_0 *with respect to* f is the winding number of x_0 about the map f.

Theorem 35. *(Lefschetz-Hopf Fixed Point Theorem [20]) If the number of fixed points of a continuous map $f : X \to X$ is finite, then*

$$\sum_{x \in Fix(f)} i(f, x) = L(f)$$

where $i(f, x)$ denotes the index of x, namely the sum of indices of fixed points of f is equal to $L(f)$.

2.3.2 CW-complexes

We will use this concept in Subsection 2.5 to define what an Euler characteristic is.

- Let X^0 be a discrete space of points. Call the elements of X^0 as $0-$*cells*.
- Let D^n_α be homeomorphic to $n-$dimensional disk D^n. Attach $n-$cells D^n_α to X^{n-1} via the continuous maps $\phi^n_\alpha : \partial D^n_\alpha \to X^{n-1}$. That is,

$$X^n = X^{n-1} \bigcup_{\phi^n_\alpha(x) \sim x} \{D^n_\alpha\}.$$

  This will give us X^n endowed with the quotient topology.

For the index set $I = \mathbb{N}$ or $I = \{0, 1, \ldots, n\}$, consider the set $X = \bigcup_{k \in I} X^k$.

The topology on X under which a subset of X is *open* (respectively *closed*) if and only if $A \cap X^n$ is *open* (respectively *closed*) for all n is called the *weak topology*.

The union $X = \bigcup_{k \in I} X^k$ endowed with with weak topology is called a *CW-complex*. The space X^k is called the *k-skeleton of X*.

For a CW-complex X,

- if $X = X^k$ for some $k \in \mathbb{N}$, then X is $k-$dimensional,
- if X is not $k-$dimensional for every $k \in \mathbb{N}$, then X is infinite dimensional.

2.4 Hyperbolic Geometry

There are several models for the hyperbolic plane. We are going to use only two of them which are the *upper half plane model* and *disk model*.

2.4.1 Upper Half Plane Model

For the rest of the thesis let $\mathbb{H}^2$:=$\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$, and call it the *upper half plane*.

In the upper half-plane model of the hyperbolic plane, the lines are of the following forms (see Figure 2.4):

- intersection of a circle centered on the $x-$axis with $\mathbb{H}^2$,

- intersection of a line perpendicular to $x-$axis with $\mathbb{H}^2$.

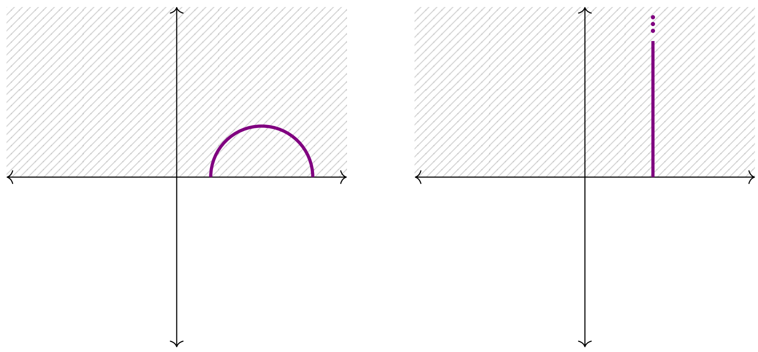

Figure 2.4: Two types of lines in the upper half plane model of hyperbolic plane

In this setting, two lines are said to be *parallel* if they do not intersect. The lines that intersect at infinity are parallel as well.

Remark 36. *If we have two points one of which is fixed, as the other point approaches $x-$axis, the distance between them tends to infinity.*

2.4.2 Disk Model

Let $\mathbb{D}^2$ denote the open unit circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$.

In the disk model of the hyperbolic plane, the lines are of the form (see Figure 2.5):

- the intersection of $\mathbb{D}^2$ with a circle $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = r \text{ for some } r \in \mathbb{R}\}$ which intersects the boundary of $\mathbb{D}^2$ perpendicularly

- a line passing through the center of the unit circle $\mathbb{D}^2$.

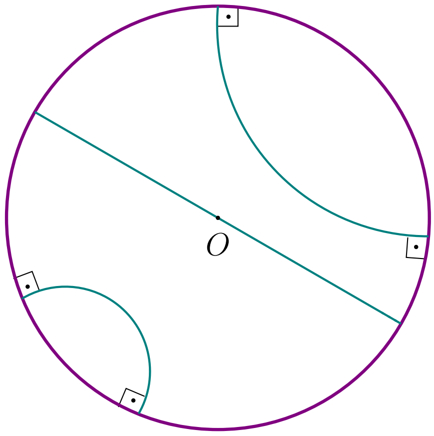

Figure 2.5: Lines in disk model of the hyperbolic plane
Here, O denotes the center of $\mathbb{D}^2$

Remark 37. *If we have two points one of which is fixed, as the other point approaches the boundary of $\mathbb{D}^2$, the distance between them approaches infinity.*

Definition 38. *A map $f : U \to V$ preserving the angle between any two curves passing through a point u for all $u \in U$ is called a* conformal map.

An invertible transformation $f(z)$ which is of the form $\frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$ is called a linear fractional transformation.

There is a conformal homeomorphism between $\mathbb{H}^2$ and $\mathbb{D}^2$.

Note that a linear fractional transformation is a conformal mapping. It can be expressed as a composition of dilations, inversions, rotations, and translations, and it maps circles or lines to circles or lines.

2.5 Euler Characteristic and Euler Class

Definition 39. *Let X be a finite-dimensional CW-complex, and let $\mathrm{cell}_k(X)$ denote the number of $k-$dimensional cells of X. Then, the Euler characteristic $\chi(X)$ of X is $\sum_{i\in\mathbb{N}}(-1)^i\mathrm{cell}_i(X)$.*

In the special case of a $2-$dimensional CW-complex, the above definition turns out *that:*

$$\chi(X) = V - E + F$$

where V, E, and F stand for the numbers of vertices, edges, and faces of X respectively.

Definition 40. *Let B be a surface. Let η be an $2-$plane bundle with total space E, base space B and projection map p. The* Euler class (or Euler number) *of η is the cohomology class $e(\eta) \in H^2(B;\mathbb{Z})$ which corresponds to $u|_E$ under the canonical isomorphism*

$$p^* : H^2(B;\mathbb{Z}) \to H^2(E;\mathbb{Z})$$

where u is the fundamental cohomology class*, i.e., the unique class such that for each fiber $F = p^{-1}(b)$, the restriction $u|_F \in H^2(F;\mathbb{Z})$ is the unique non-zero class in $H^2(F;\mathbb{Z})$.*

Remark 41. *Euler characteristic and Euler class are preserved under homotopy equivalence.*

In 1970, Wood [9] proved the following inequality.

Theorem 42. *([21], Milnor-Wood Inequality) A flat orientable circle bundle $\widehat{E}_\rho$ over a closed surface Σ_g of genus g has the Euler number satisfying $|e(\widehat{E}_\rho)| \leq 2g - 2$.*

This was a generalization of a previous result shown by Milnor [3] in 1957. The following theorem is equivalent to Milnor's result.

Theorem 43. *([21], Milnor's Inequality) A flat linear orientable circle bundle E_ρ over a closed surface Σ_g of genus g has the Euler number satisfying $|e(E_\rho)| \leq g - 1$.*

As referred in [21], [10] gives a proof of the following special case of Milnor-Wood Inequality:

Lemma 44. *[21] For a closed orientable hyperbolic surface Σ_g of genus $g \geq 2$, the circle bundle corresponding to the induced action of the group of deck transformations of the upper half plane $\mathbb{H}^2$ on $\partial_\infty \mathbb{H}^2$ has Euler number $2 - 2g$.*

CHAPTER 3

MAIN RESULTS

3.1 Some Groups of Mapping Classes Not Realized By Diffeomorphisms

Throughout this section, we will use the map F which is described in the following subsection.

3.1.1 The map F

Let $z \in \Sigma_g$ be a point. Remember the Birman exact sequence

$$1 \to \pi_1(\Sigma_g, z) \xrightarrow{F} \mathrm{Mod}(\Sigma_g, z) \to \mathrm{Mod}(\Sigma_g) \to 1.$$

Here, to understand the homomorphism $F : \pi_1(\Sigma_g, z) \hookrightarrow \mathrm{Mod}(\Sigma_g, z)$, let $\gamma \in \pi_1(\Sigma_g, z)$, and let $\overrightarrow{\gamma} : [0,1] \to \Sigma_g$ be an element of the homotopy class of γ, the map F pushes z along the path $\overrightarrow{\gamma} \in \gamma$, and drag the rest of the surface as z goes (see Figure 3.1). Birman called this map as *spin map* [19]. Note that $F(\gamma) \in \mathrm{Mod}(\Sigma_g, z)$.

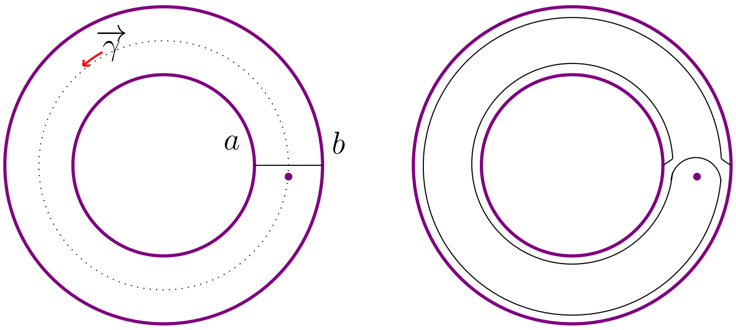

Figure 3.1: How the map F behaves

The map sending t to $\overrightarrow{\gamma}(1-t)$ can be considered as an isotopy from $id_{\{z\}}$ to itself. As stated in Section 3 of [21], there is an isotopy $f_t : \Sigma_g \to \Sigma_g$ such that $f_t(z) = \overrightarrow{\gamma}(1-t)$ and $f_0 = id_{\Sigma_g}$.

3.1.2 Needed Theorems

We will need the following lemma to prove Proposition 46. For this reason, we need the setting that we will use in Proposition 46.

Recall that the group $\mathcal{G}(\Sigma_g, z)$ is the set of all orientation preserving homeomorphisms $f : \Sigma_g \to \Sigma_g$ such that z is a fixed point of f and both f, and f^{-1} are differentiable at z. There is an epimorphism $\mathcal{G}(\Sigma_g, z) \to \mathrm{Mod}(\Sigma_g, z)$.
Assume that there is a realization:

$$\begin{tikzcd} & \mathcal{G}(\Sigma_g, z) \arrow[d, two heads] \\ \pi_1(\Sigma_g, z) \arrow[ur, dashed, "\phi"] \arrow[r, "F"] & \mathrm{Mod}(\Sigma_g, z). \end{tikzcd}$$

We can consider $\mathbb{H}^2$ as the universal cover of Σ_g by endowing Σ_g with a hyperbolic metric.

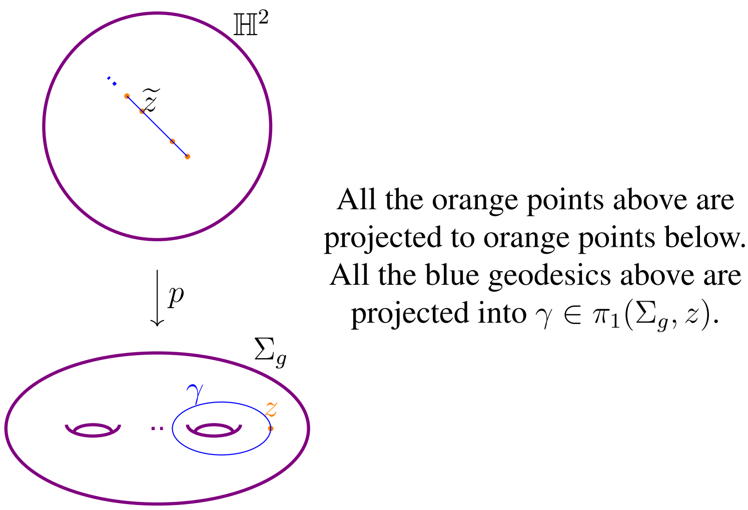

Figure 3.2: $\mathbb{H}^2$ as the universal cover of Σ_g

Choose a point $\widetilde{z} \in p^{-1}(z)$, and a representative ϕ_γ from $F(\gamma)$. By Homotopy Lifting

Property 32, there is a lift of ϕ_γ which is unique by Unique Lifting Property 31. Let $\widetilde{\phi} : \pi_1(\Sigma_g, z) \to \mathcal{G}(\mathbb{H}^2, \widetilde{z})$ be the map sending γ to this unique lift. With this setting, we are ready to state the following lemma.

Lemma 45. *([21], Lemma 3.2) For $\gamma \in \pi_1(\Sigma_g, z)$, the homeomorphism $\widetilde{\phi}_\gamma = \widetilde{\phi}(\gamma) : \mathbb{H}^2 \to \mathbb{H}^2$ can be extended to the closure $\overline{\mathbb{H}}^2 = \mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$ of $\mathbb{H}^2$. Moreover, the action of this extended homeomorphism on $\partial \mathbb{H}^2$ is the same as the action of deck transformation $\overline{\gamma}$ corresponding to γ.*

Proof. By definition of F (see Subsection 3.1.1), we know that if the point z were not marked, then $F(\gamma)$ would be trivial. This means, ϕ_γ would be homotopic to the identity in this case. Assume the point z is not a marked point. Remember the map f_t in Subsection 3.1.1. We get f_t is a homotopy from the identity to ϕ_γ with $f_0 = id_{\Sigma_g}$ and $f_1 = \phi_\gamma$. By Unique Lifting Property 31, there is a unique lift $\widehat{f_t}$ of f to $\mathbb{H}^2$ with $\widehat{f_0} = id_{\mathbb{H}^2}$. This gives us a new lift $\widehat{\phi_\gamma} := \widehat{f_1}$ of ϕ_γ. In other words, ϕ could be lifted in a different way.

We get $\widehat{\phi_\gamma}(\widetilde{z}) = \overline{\gamma}^{-1}\widetilde{z}$ where $\overline{\gamma}$ is the deck transformation corresponding to γ. By Unique Lifting Property 31, knowing the image at $\widetilde{z}$ tells us the whole lifting. For this reason,

$$\overline{\gamma} \circ \widehat{\phi_\gamma} = \widetilde{\phi}_\gamma. \tag{3.1}$$

Therefore, $\widehat{\phi}$ moves every point in the hyperbolic plane $\mathbb{H}^2$ by at most a fixed constant. This makes the distance between a point $p \in \mathbb{H}^2$ and its image $\widehat{\phi}(p)$ smaller and smaller as we get near to the boundary at infinity $\partial_\infty \mathbb{H}^2$ of the hyperbolic plane. Thus, $\widetilde{\phi}$ continuously extends to $\partial_\infty \mathbb{H}^2$, and its action on $\partial_\infty \mathbb{H}^2$ is the same as identity. By (3.1), the action of $\widetilde{\phi}_\gamma$ coincides with $\overline{\gamma}$. □

We will need the following proposition, which implies that the surface braid group $B_n(\Sigma_g)$ does not lift to $\mathcal{G}(\Sigma_g, z)$, and so does not lift to $\mathrm{Diff}^+(\Sigma_g, \mathbf{z})$ as well, to prove Theorem 52.

Proposition 46. *([21], Proposition 3.1) For a closed surface Σ_g of genus $g \geq 2$ with a marked point $z \in \Sigma_g$, the inclusion F (see Subsection 3.1.1) of a finite index subgroup $\Gamma \leq \pi_1(\Sigma_g, z)$ does not lift to $\mathcal{G}(\Sigma_g, z)$.*

Proof. To get a contradiction, assume that there is a realization $\phi : \pi_1(\Sigma_g, z) \to \mathcal{G}(\Sigma_g, z)$. For an element $\gamma \in \pi_1(\Sigma_g, z)$, there is an element $f_1 \in \mathrm{Diff}^+(\Sigma_g, z)$ (as described in Subsection 3.1.1) which depends only on γ, and there is a mapping class $F(\gamma) \in \mathrm{Mod}(\Sigma_g, z)$ (as described in Subsection 3.1.1). Recall the setting before Lemma 45, as explained there, there is a unique lift $\widetilde{\phi}_\gamma$ of ϕ_γ, and so a homomorphism $\widetilde{\phi} : \pi_1(\Sigma_g, z) \to \mathcal{G}(\mathbb{H}^2, \widetilde{z})$ sending γ to $\widetilde{\phi}_\gamma$.

Recall that $\overline{\mathbb{H}}^2$ is the union of the upper half plane, the $x-$axis and the point at infinity. This means we can consider $\overline{\mathbb{H}}^2 \setminus \{\widetilde{z}\}$ as a half-open annulus. To this open side of the annulus, attach the space of directions $P_+T_{\widetilde{z}}\mathbb{H}^2$ of the tangent space at $\widetilde{z}$, and name the resulting closed annulus $\mathcal{A}$ (see Figure 3.3).

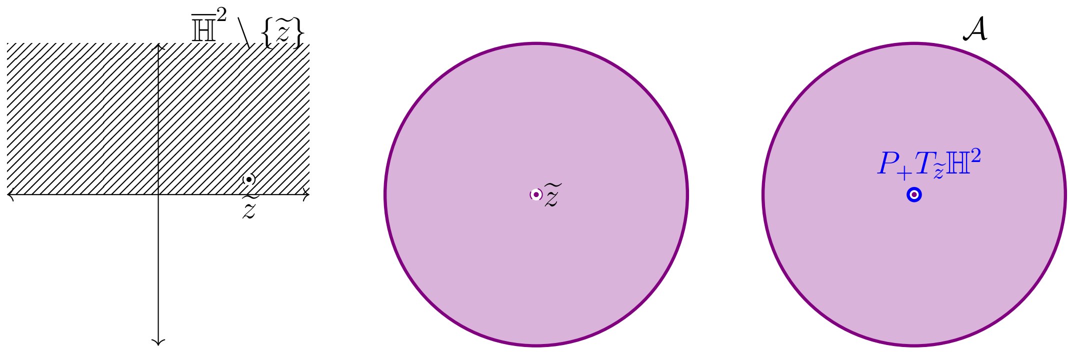

Figure 3.3: How we get the closed annulus $\mathcal{A}$ from $\overline{\mathbb{H}}^2 \setminus \{\widetilde{z}\}$.
The blue circle represents the space of directions of the tangent space at $\widetilde{z}$.

By Lemma 45, the map $\widetilde{\phi}_\gamma$ extends to $\overline{\mathbb{H}}^2$ for all $\gamma \in \pi_1(\Sigma_g, z)$, which leads us to an action of $\pi_1(\Sigma_g, z)$ on $\overline{\mathbb{H}}^2 \setminus \{\widetilde{z}\}$. This means we have an action on the closed annulus $\mathcal{A}$ except on its boundary component which is made by attaching the space of directions. We can extend this action to whole annulus $\mathcal{A}$ because $\widetilde{\phi}_\gamma$ is differentiable at $\widetilde{z}$ for all $\gamma \in \pi_1(\Sigma_g, z)$, and so we already have an action on the space of directions which is induced by the map $\gamma \mapsto d\widetilde{\phi}_\gamma|_{\widetilde{z}}$ from $\pi_1(\Sigma_g, z)$ to $\mathrm{GL}^+(T_{\widetilde{z}}\mathbb{H}^2)$. Here, we used the map $\widetilde{\phi}_\gamma|_{\widetilde{z}}$ because in order to have the image of the tangent space around the point $\widetilde{z}$ a tangent space around $\widetilde{z}$ again, we need a map fixing the point $\widetilde{z}$.

By Milnor's Inequality 43, the circle bundle E_1 over Σ_g induced by this action has Euler number $|e(E_1)| \leq g - 1$. On the other boundary component, the action is, as described in Lemma 45, induced by deck transformations. By Lemma 44, the circle bundle E_2 induced by this action has Euler number $|e(E_2)| = 2 - 2g$.

There is a deformation retraction from $\mathcal{A}$ to E_1 and a deformation retraction from $\mathcal{A}$ to E_2. For this reason, the Euler numbers $e(E_1)$ and $e(E_2)$ should be the same by the Remark 41, which implies $2 \cdot |1-g| = |2-2g| = |e(E_2)| = |e(E_1)| \leq g-1$, but this cannot happen. Therefore, there cannot exist a lift ϕ. For the case of a finite index subgroup, the argument is the same. $\square$

We will need the following lemma to prove Theorem 52 Part b).

Lemma 47. *([21], Lemma 4.1) By means of the projection map* $p_1 : \mathrm{Conf}_k(\Sigma_g) \to \Sigma_g$ *defined as*

$$(x_1, x_2, \ldots, x_k) \mapsto x_1,$$

the configuration space $\mathrm{Conf}_k(\Sigma_g)$ *becomes a fiber bundle over* Σ_g.

Let $\pi_1(p_1) : \pi_1(\mathrm{Conf}_k(\Sigma_g), (z_1, z_2, \ldots, z_k)) \to \pi_1(\Sigma_g, z_1)$ *be the homomorphism resulted from taking the related part of the long exact sequence of homotopy groups. Then, this map has a right inverse* $\eta : \pi_1(\Sigma_g, z_1) \to \pi_1(\mathrm{Conf}_k(\Sigma_g), (z_1, z_2, \ldots, z_k))$.

$$\begin{tikzcd} \mathrm{Conf}_k(\Sigma_g) \arrow[d, "p_1"'] \\ \Sigma_g \arrow[u, bend right, dotted, "\sigma"'] \end{tikzcd}$$

Figure 3.4: The section σ

Proof. Consider a compact $T \subseteq \Sigma_g$ surface homeomorphic to a torus with one boundary component, and an essential (i.e., not homotopic to a marked point or a boundary component) simple closed curve $C \subseteq T$ with $z_i \in C$ for all $i \in \{1, 2, \ldots, k\}$. Identify the remaining part, $\Sigma_g \setminus \mathrm{int}(T)$ with a point, and call the resulting torus $\mathbb{T}$. Project this torus to C. Figure 3.5 shows this process.

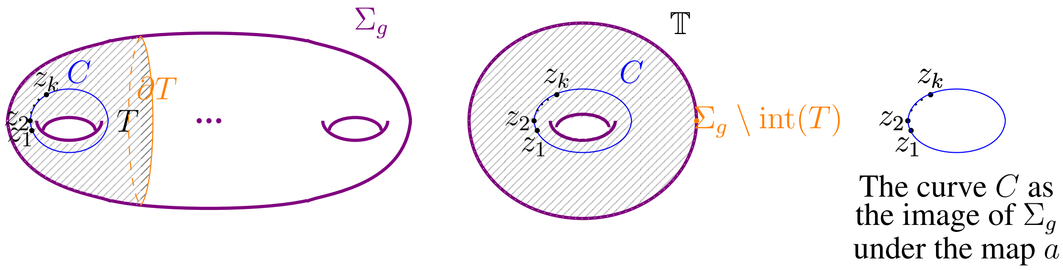

Figure 3.5: The map $a : \Sigma_g \to C$

In this way, we get a map $a : \Sigma_g \to C$ fixing the curve C pointwise. Let r_i be the rotation of C sending z_1 to z_i, and let $\iota : C \hookrightarrow \Sigma_g$ be the inclusion map. Define $\alpha_i : \Sigma_g \to \Sigma_g$ to be the composition $\iota \circ r_i \circ a$.

Fact 48. *The followings hold.*

- *The image of C under the map α_i is C for all $i \in \{1, 2, \dots, k\}$ by definition.*
- *$\alpha_i(z_1) = z_i$ for all $i \in \{1, 2, \dots, k\}$ by definition.*
- *The restriction of α_1 to C is the identity map by its definition.*
- *$\alpha_i : \Sigma_g \to \Sigma_g$ has no fixed points if $i \neq 1$. Because*
  - *If a point $z \in \Sigma_g$ is not in the curve C, then since it is mapped to C, its image cannot be z anymore.*
  - *If the point $z \in \Sigma_g$ is in C, then it is first mapped to itself by the map a, and then rotated which makes the image of z different than itself again.*
- *For all $x \in \Sigma_g$, $\alpha_i(x) \neq \alpha_j(x)$ if $i \neq j$ because after mapping the surface Σ_g to the curve C via the map a, the rotations r_i and r_j rotate the point x in different fashions.*

Let $\sigma(x) := (x, \alpha_2(x), \dots, \alpha_k(x))$ be a map.

- The image of σ is in fact the configuration space $\mathrm{Conf}_k(\Sigma_g)$ because there are no two components $\alpha_i(x), \alpha_j(x)$ with $i \neq j$ and $\alpha_i(x) = \alpha_j(x)$ for some $x \in \Sigma_g$ by Fact 48.

- $p_1(\sigma(x)) = x$ by definition.

This makes the map $\sigma : \Sigma_g \to \mathrm{Conf}_k(\Sigma_g)$ a section of the fiber bundle

$$p_1 : \mathrm{Conf}_k(\Sigma_g) \to \Sigma_g.$$

Then, there is a homomorphism

$$\eta : \pi_1(\Sigma_g, z_1) \to \pi_1(\mathrm{Conf}_k(\Sigma_g), (z_1, z_2, \dots, z_k))$$

induced by the map σ with $\pi_1(p_1) \circ \eta$ is the identity map. $\square$

Proposition 49. *[21] For a surface Σ_g of genus $g \geq 6$, there is a diffeomorphism $\tau : \Sigma_g \to \Sigma_g$ of order $o(\tau) = 3$ whose number of fixed points $|\mathrm{Fix}(\tau)|$ is greater than or equal to 2 such that the quotient $\Sigma_g/\langle\tau\rangle$ has genus $h \geq 2$.*

Proof. For a proof given in a more general way, see the Fact of Case 3 in the proof of Theorem 1.2 in [21]. We will prove the proposition by using two different ways.

Case 1: $g = 3h + 2$ for some integer $h \geq 2$

Let us identify the $xy-$plane with the complex plane $\mathbb{C}$, let T be the equilateral triangular region in this plane with vertices $v_k = e^{i\frac{2\pi}{3}k}$ for $k \in \{1, 2, 3\}$. Let us identify the $z = 5$ plane with the complex plane $\mathbb{C}$ and consider the triangular region with vertices $\overline{v}_k = e^{i\frac{2\pi}{3}k}$ for $k \in \{1, 2, 3\}$. For $k \in \{1, 2, 3\}$, consider the line segments $v_k\overline{v}_k$ between the vertices of these two triangular regions. For all $i \in \{1, 2, 3\}$ and for all $j \in \{1, 2, \dots, h\}$, add the arcs l^i_j starting at $\overline{v}_i$ ending at v_i which intersect the line segment $v_k\overline{v}_k$ only at the points $\overline{v_i}$, v_i, and which gives for any two $j_1, j_2 \in \{1, 2, \dots, h\}$, the intersection $l^i_{j_1} \cap l^i_{j_2} = \{\overline{v_i}, v_i\}$. Call the resulting object $\overline{\Sigma}_g$. We can choose l^i_j so that $\overline{\Sigma}_g$ is invariant under the rotation $\overline{\tau}$ by $\frac{2\pi}{3}-$radians about $z-$axis as it can be seen in the Figure 3.6. Let Σ_g be the boundary of a tubular neighborhood of $\overline{\Sigma}_g$. The diffeomorphism $\overline{\tau}$ induces a diffeomorphism $\tau : \Sigma_g \to \Sigma_g$ of order 3 with 4 fixed points, and $\Sigma_g/\langle\tau\rangle$ is an orientable surface of genus $h \geq 2$ as we needed.

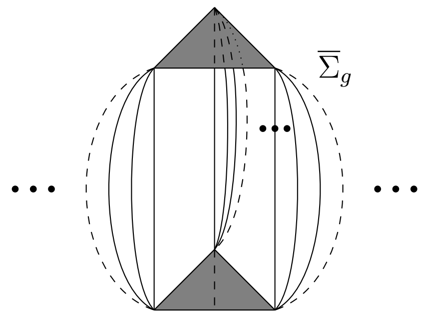

Figure 3.6: $\overline{\Sigma}_g$ in Case 3

Case 2: $g = 3h + 1$ *for some integer* $h \geq 2$
Let $h \neq 2$. Take $\overline{\Sigma}_g$ in the previous case. Identify $z = -5$ plane with the complex pane $\mathbb{C}$ and consider the equilateral triangular region whose vertices are $\widehat{v}_k = e^{i\frac{2\pi}{3}k}$ for $k \in \{1, 2, 3\}$. Draw the line segments $\overline{v}_k\widehat{v}_k$ for $k \in \{1, 2, 3\}$. Call the resulting object $\widehat{\Sigma}_g$ (see Figure 3.7). Define τ same as the case before. Take the boundary of a regular neighborhood of $\widehat{\Sigma}_g$ say Σ_g. Similar to the case before, Σ_g is invariant under τ, τ has 6 fixed points on Σ_g, and $\Sigma_g/\langle\tau\rangle$ is an orientable surface of genus $h > 2$.

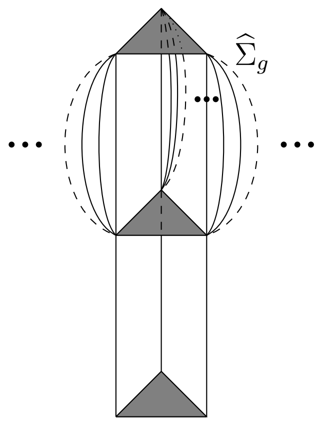

Figure 3.7: $\widehat{\Sigma}_g$ in Case 2, $x \neq 2$

To prove the case $g = 7$, consider a regular $30-$gonal region T as in Figure 3.8. Identify its antipodal edges to get a genus-7 surface Σ_7. Define the diffeomorphism $\widehat{\tau}$ on T as the rotation by $\frac{2\pi}{3}$ about the center O of T. Then, $\widehat{\tau}$ is of order 3. $\widehat{\tau}(x_1) = x_1$, $\widehat{\tau}(x_2) = x_2, \widehat{\tau}(O) = O$ and these are the only fixed points of $\widehat{\tau}$. $\widehat{\tau}$ induces a diffeomorphism $\tau : \Sigma_g \to \Sigma_g$ which has 3 fixed points, the images of O, x_1 and x_2 under $T \to \Sigma_g$.

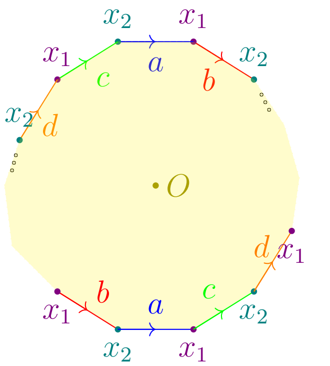

Figure 3.8: Regular 30-gonal region T

$\Sigma_7 \to \Sigma_7/\langle\tau\rangle$ is a $3-1$ branched cover branching on 3 points. Then,

$$\chi(\Sigma_7 \setminus \{x_1, x_2, O\}) = 3\chi(\Sigma_7/\langle\tau\rangle \setminus \{\tau(x_1), \tau(x_2), \tau(O)\}).$$

Say $\Sigma_g/\langle\tau\rangle$ is a genus-k surface. We have $2 - 2\cdot 7 - 3 = 3(2 - 2k - 3)$ which gives the fact that $\Sigma_7/\langle\tau\rangle$ is a genus-2 surface.

Case 3: $g = 3h$ for some integer $h \geq 2$

Similar to Case 2 before, consider the $4g-$gonal region P whose edges are identified as in the Figure 3.9.

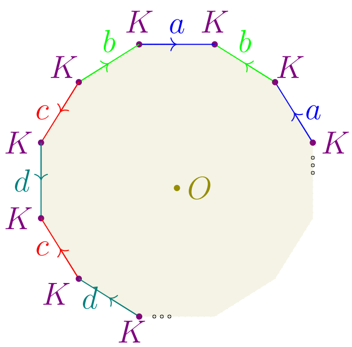

Figure 3.9: Regular $4g-$gonal region P

This gives us a genus-g surface Σ_g. Let $\widehat{\tau} : P \to P$ be the diffeomorphism defined as the rotation by $\frac{2\pi}{3}$ radians about O. Then, the only fixed points of $\widehat{\tau}$ on P are O and K. $\widehat{\tau}$ induces a diffeomorphism $\tau : \Sigma_g \to \Sigma_g$ which has 2 fixes points which are the images of the fixed points of $\widehat{\tau}$ under the map $P \to \Sigma_g$.

$\Sigma_g \to \Sigma_g/\langle\tau\rangle$ is a $2-1$ branched cover branching at 2 points. Then,

$$\chi(\Sigma_g \setminus \{K, O\}) = 2\chi(\Sigma_g/\langle\tau\rangle \setminus \{\tau(K), \tau(O)\}).$$

Let $\Sigma_g/\langle\tau\rangle$ be of genus k. Then, $2-2g-2 = 2(2-2k-2)$ which gives $g = 2k$. Since $g > 6$, $\Sigma_g/\langle\tau\rangle$ has genus $k \geq 2$. $\square$

Lemma 50. *([21], Remark of Case 3 of Proof of Theorem 1.2) For the diffeomorphism $\tau : \Sigma_g \to \Sigma_g$ mentioned in Proposition 49, and the mapping class T corresponding to τ, if we assume that the map ψ from the centralizer $C(T)$ to $\mathrm{Diff}^+(\Sigma_g)$ is a lifting, then $\psi(T)$ and τ are conjugate.*

Let $C(\tau, \mathbf{z})$ denote the subgroup of $C(\tau)$ whose elements fix the set $\mathbf{z}$ pointwise.

Lemma 51. *([21], Lemma 4.2) For the diffeomorphism τ and $\overline{\mathbf{z}}$ the set of projections of the fixed points of τ, the homomorphism α given by the homeomorphisms of $\Sigma_g/\langle\tau\rangle$ fixing $\overline{\mathbf{z}}$ induced by the diffeomorphisms $f \in C(\tau)$ (for more details about these, see the proof of Part (a) of Theorem 52), the image $\alpha(C(\tau, \mathbf{z}))$ under α of the subgroup $C(\tau, \mathbf{z})$ is a subset of $\mathcal{G}(\Sigma_g/\langle\tau\rangle, \overline{\mathbf{z}})$.*

Proof. There is a conformal structure which makes τ biholomorphic. For a fixed point x of τ, since τ is of order 3, there are coordinates ζ around x with $\tau(\zeta) = \omega \cdot \zeta$ where ω is the third root of unity. Note that by $o(\omega) = 3$, $\{1, \omega\}$ span $\mathbb{C}$ as a real vector space. Let $f \in C(\tau, \mathbf{z})$, consider the derivative $df_x : T_x\Sigma_g \to T_x\Sigma_g$ of f around x. Since $f(\tau(\zeta)) = \tau(f(\zeta))$, $f(\omega\zeta) = \omega(f(\zeta))$. So, $df_x\omega = \omega df_x$. This means df_x commutes with any element of $\mathbb{C}$ as $\{1, \omega\}$ spans $\mathbb{C}$. So, it is differentiable. Then, considering the projections of the surface Σ_g and its tangent space around the point x, and considering the compositions with the projection p to $\Sigma_g/\langle\tau\rangle$ of the maps f and df_x, we get the derivative $d\alpha(f)_{p(x)}$ of the map $\alpha(f) : \Sigma_g/\langle\tau\rangle \to \Sigma_g/\langle\tau\rangle$ induced by f at the projection of the point x. This means $\alpha(f)$ is differentiable with a differentiable inverse at $p(x)$ satisfying $\alpha(f)(p(x)) = p(x)$. Since x was an arbitrary fixed point, this property of $\alpha(f)$ holds for any $p(x) \in \overline{\mathbf{z}}$. Thus, $\alpha(C(\tau, \mathbf{z})) \subseteq \mathcal{G}(\Sigma_g/\langle\tau\rangle, \overline{\mathbf{z}})$. $\square$

3.1.3 Morita's Non-lifting Theorem

In his paper [13], Morita stated the following theorem for the surface Σ_g of genus $g \geq 18$, and later in [16], he showed that $g \geq 5$ is enough to get the same result instead of 18. Here, we will prove the cases for $g \geq 6$ with an arbitrary number of marked points and $g \geq 2$ with at least one marked point in a different way.

Theorem 52. *([21], Theorem 1.2) Let Σ_g be a surface of genus g, $\mathbf{z}$ be the set of marked points of Σ_g. Then,*
(a) If $g \geq 6$, the exact sequence

$$0 \to \mathrm{Diff}^0(\Sigma_g, \mathbf{z}) \to \mathrm{Diff}^+(\Sigma_g, \mathbf{z}) \to \mathrm{Mod}(\Sigma_g, \mathbf{z}) \to 0$$

does not split.
(b) If $g \geq 2$ and the cardinality $|\mathbf{z}| \geq 1$, then the above result holds. Moreover, no finite index subgroup of $\mathrm{Mod}(\Sigma_g, \mathbf{z})$ lifts to $\mathrm{Diff}^+(\Sigma_g, \mathbf{z})$. (Also, no finite index subgroup of $\mathrm{Mod}(\Sigma_g, \mathbf{z})$ lifts to $\mathcal{G}(\Sigma_g, z_1)$.)

Proof. (b) Note that the group $\mathrm{Diff}^+(\Sigma_g, z)$ is a subgroup of $\mathcal{G}(\Sigma_g, z)$, so the group $\mathrm{Mod}(\Sigma, z)$ does not lift to $\mathrm{Diff}^+(\Sigma_g, z)$ by Proposition 46. This proves the case for $|\mathbf{z}| = 1$ and $g \geq 2$.

Assume $g \geq 2, |\mathbf{z}| \geq 2$. By means of the projection map $p_1 : \mathrm{Conf}_k(\Sigma_g) \to \Sigma_g$ defined as $(x_1, x_2, \ldots, x_k) \mapsto x_1$, the configuration space $\mathrm{Conf}_k(\Sigma_g)$ becomes a fiber bundle over Σ_g. By taking the related part of the long exact sequence of homotopy groups, we get a homomorphism

$$\pi_1(p_1) : \pi_1(\mathrm{Conf}_k(\Sigma_g), (z_1, z_2, \ldots, z_k)) \to \pi_1(\Sigma_g, z_1).$$

By Lemma 47, we know that this homomorphism has a right inverse. Let

$$\vec{\mathbf{z}} := (z_1, z_2, \ldots, z_k) \in \mathrm{Conf}_k(\Sigma_g).$$

Forgetting all marked points except z_1 gives the same exact sequence as we have seen in Subsection 3.1.1 before, and if we forget all the marked points, the Birman Exact Sequence (see Theorem 7) becomes:

$$1 \to \pi_1(\mathrm{Conf}_k(\Sigma_g), \vec{\mathbf{z}}) \to \mathrm{PMod}(\Sigma_g, \mathbf{z}) \to \mathrm{Mod}(\Sigma_g) \to 1.$$

Considering these sequences with the maps η and $\pi_1(p_1)$ before, we get:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\mathrm{Conf}_k(\Sigma_g), \vec{\mathbf{z}}) \arrow[r] \arrow[d, "\pi_1(p_1)"] & \mathrm{PMod}(\Sigma_g, \mathbf{z}) \arrow[r] \arrow[d] & \mathrm{Mod}(\Sigma_g) \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & \pi_1(\Sigma_g, z_1) \arrow[r] \arrow[u, bend left, "\eta"] & \mathrm{Mod}(\Sigma_g, z_1) \arrow[r] & \mathrm{Mod}(\Sigma_g) \arrow[r] & 1.
\end{tikzcd}$$

In particular, the map between groups $\pi_1(\mathrm{Conf}_k(\Sigma_g), \vec{\mathbf{z}})$ and $\mathrm{PMod}(\Sigma_g, \mathbf{z})$ is injective. Let $G \leq \mathrm{Mod}(\Sigma_g, \mathbf{z})$ be a finite index subgroup. Then, $\mathrm{PMod}(\Sigma_g, \mathbf{z}) \cap G$ is a finite index subgroup of $\mathrm{PMod}(\Sigma_g, \mathbf{z})$. Let Γ be the inverse image of $\mathrm{PMod}(\Sigma_g, \mathbf{z}) \cap G$ in $\pi_1(\mathrm{Conf}_k(\Sigma_g), \vec{\mathbf{z}})$. By the map between $\pi_1(\mathrm{Conf}_k(\Sigma_g), \vec{\mathbf{z}})$ and $\mathrm{PMod}(\Sigma_g, \mathbf{z})$ being injective, Γ has finite index in $\pi_1(\mathrm{Conf}_k(\Sigma_g), \vec{\mathbf{z}})$. Then, the image $\pi_1(p_1)(\Gamma)$ of the group Γ under the function $\pi_1(p_1)$ has finite index in $\pi_1(\Sigma_g, z_1)$. Since there exists a right inverse η of the map $\pi_1(p_1)$, if the group G lifts to $\mathrm{Diff}^+(\Sigma_g, \mathbf{z})$, then $\pi_1(p_1)(\Gamma)$ would lift to $\mathrm{Diff}^+(\Sigma_g, \mathbf{z})$. But this cannot happen by Proposition 46, and by the fact that $\mathrm{Diff}^+(\Sigma_g, z_1) \leq \mathcal{G}(\Sigma_g, z_1)$.

(a) By Part b), the case of $g \geq 6$ and $|\mathbf{z}| \geq 1$ follows. So, assume $|\mathbf{z}| = 0$. By Proposition 49, there is a diffeomorphism $\tau : \Sigma_g \to \Sigma_g$ of order 3 such that the number of fixed points of τ is greater than or equal to 2 and that $\Sigma_g/\langle\tau\rangle$ is of genus at least 2. Assume that the centralizer $C(T)$ lifts to $\mathrm{Diff}^+(\Sigma_g)$ via the map $\psi : C(T) \to \mathrm{Diff}^+(\Sigma_g)$, where T is the class of the diffeomorphism τ in $\mathrm{Mod}(\Sigma_g)$. Then, the order of $\psi(T)$ is 3. Since T is the mapping class corresponding to τ and since ψ is a lifting, $\psi(T)$ and τ are isotopic. By Lemma 50, these two are conjugate. So, we may assume without loss of generality that $\psi(T) = \tau$.

Let $p : \Sigma_g \to \Sigma_g/\langle\tau\rangle$ be the projection map, and let $\overline{\mathbf{z}} = \{\overline{z_1}, \overline{z_2}, \ldots, \overline{z_k}\}$ be the set of projections of the fixed points of Σ_g to $\Sigma_g/\langle\tau\rangle$. Any $f \in C(\tau)$ induces a map $K : \Sigma_g/\langle\tau\rangle \setminus \overline{\mathbf{z}} \to \Sigma_g/\langle\tau\rangle \setminus \overline{\mathbf{z}}$ with $x, \tau^k(x) \in \Sigma_g \setminus Fix(\tau)$ giving the same image under K for $k \in \mathbb{Z}$. Extending K gives us a homeomorphism K of $\Sigma_g/\langle\tau\rangle$ fixing $\overline{\mathbf{z}}$. Thus, we can define a homomorphism $\alpha : C(\tau) \to \mathrm{Homeo}^+(\Sigma_g/\langle\tau\rangle, \overline{\mathbf{z}})$ defined as $\alpha(f) = K$. By definition of α, we have $\ker(\alpha) = \langle\tau\rangle$. We had $\psi : C(T) \to C(\tau)$ before. By Lemma 51, $\alpha(C(\tau, \mathbf{z})) \subseteq \mathcal{G}(\Sigma_g/\langle\tau\rangle, \overline{\mathbf{z}})$. Since $\psi(T) = \tau$, we have $\psi : C(T)/\langle T\rangle \to C(\tau)/\langle\tau\rangle$. Since $\langle\tau\rangle$ is the kernel of α, by considering the composition $\alpha \circ \psi$, we get an action $C(T)/\langle T\rangle \to \mathrm{Homeo}^+(\Sigma_g/\langle\tau\rangle, \overline{\mathbf{z}})$.

Via the lifting ψ, we can identify $C(T)/\langle T\rangle$ with $C(\tau)/\langle\tau\rangle$. Remember from the definition of the map α that a diffeomorphism $f \in \mathrm{Diff}^+(\Sigma_g)$ commuting with τ induces a homeomorphism $\Sigma_g/\langle\tau\rangle \to \Sigma_g/\langle\tau\rangle$ fixing $\overline{\mathbf{z}}$ setwise.

Consider the image of $C(T)/\langle T\rangle$ under the map

$$C(T)/\langle T\rangle \to \mathrm{Homeo}^+(\Sigma_g/\langle\tau\rangle, \overline{\mathbf{z}}) \to \mathrm{Mod}(\Sigma_g/\langle\tau\rangle, \overline{\mathbf{z}}).$$

If the image of an element of $\mathrm{Homeo}^+(\Sigma_g/\langle\tau\rangle, \overline{\mathbf{z}})$ under the map

$$\mathrm{Homeo}^+(\Sigma_g/\langle\tau\rangle, \overline{\mathbf{z}}) \to \mathrm{Mod}(\Sigma_g/\langle\tau\rangle, \overline{\mathbf{z}})$$

is the identity, then this element is either the identity in $\mathrm{Homeo}^+(\Sigma_g/\langle\tau\rangle, \overline{\mathbf{z}})$ or τ or τ^2. But, all of these three elements correspond to the identity in $C(T)/\langle T\rangle$. So, there is a subgroup G of $\mathrm{Mod}(\Sigma_g/\langle\tau\rangle, \overline{\mathbf{z}})$ that can be identified with $C(T)/\langle T\rangle$.

Let $\Gamma{:=}G \cap \mathrm{PMod}(\Sigma_g/\langle\tau\rangle, \overline{\mathbf{z}})$. Then, by both G and $\mathrm{PMod}(\Sigma_g/\langle\tau\rangle, \overline{\mathbf{z}})$ being of finite index, Γ is of finite index in $\mathrm{Mod}(\Sigma_g/\langle\tau\rangle, \overline{\mathbf{z}})$. For each $x \in C(T), xT = Tx$, so $\psi(x)\psi(T) = \psi(T)\psi(x)$. This implies that $\psi(x) \in C(\psi(T))$. Therefore, $\psi(C(T)) \subseteq C(\psi(T)) = C(\tau)$ as we assumed $\psi(T) = \tau$. Thus, the image of Γ under ψ is contained in $C(\tau)$. Moreover, since $\Gamma \subseteq \mathrm{PMod}(\Sigma_g/\langle\tau\rangle, \overline{\mathbf{z}})$, $\psi(\Gamma)$ should be contained in $C(\tau, \mathbf{z})$. By Lemma 51, $\alpha(\psi(\Gamma))$ is a subset of $\mathcal{G}(\Sigma_g/\langle\tau\rangle, \overline{\mathbf{z}})$. This means a finite index subgroup of $\mathrm{Mod}(\Sigma_g/\langle\tau\rangle, \overline{\mathbf{z}})$ lifts to $\mathcal{G}(\Sigma_g/\langle\tau\rangle, \overline{\mathbf{z}})$ where $\Sigma_g/\langle\tau\rangle$ is a surface of genus $h \geq 2$ with at least 1 marked points which contradicts the note in parenthesis of Case *(b)* of this theorem. Therefore, there cannot be such a lift ψ. $\square$

3.2 On Non-realizability of Braid Groups By Diffeomorphisms

3.2.1 Needed Theorems

As stated in [22], Theorem 4.1 (iii) of [14] implies that:

Theorem 53. *([22], Theorem 3.6) Let $\mathbf{z}_n$ be the set of n marked points in the closed disc $\mathbb{D}^2$.*
(a) The inclusion map $(\mathbb{D}^2, \mathbf{z}_n) \hookrightarrow (\Sigma, \mathbf{z}_n)$ induces an injection $B_n \hookrightarrow B_n(\Sigma)$ where the surface Σ is not the sphere S^2.

(b) If $\Sigma = S^2$, identify B_n with $\mathrm{Mod}(\mathbb{D}^2, \mathbf{z}_n)$ via the isomorphism $B_n \cong \mathrm{Mod}(\mathbb{D}, \mathbf{z}_n)$ in Remark 15. Let Δ denote the Dehn twist of a boundary parallel curve in $\mathbb{D}^2$. Then, the inclusion $(\mathbb{D}, \mathbf{z}_n) \hookrightarrow (S^2, \mathbf{z}_{n+1})$ induces a homomorphism $\phi : B_n \to B_{n+1}(S^2)$ such that $\ker(\phi) \subseteq \langle \Delta \rangle$ and $\ker(\phi)$ is contained in the center of B_n.

Proposition 54. *([22], Proposition 3.4)*
(i) For $n \geq 5$, the set

$$S = \{\sigma_i \sigma_{i+1}^{-1} : 1 \leq i \leq n-2\}$$

generates $[B_n, B_n]$, and any two $\sigma_i \sigma_{i+1}^{-1}, \sigma_j \sigma_{j+1}^{-1} \in S$ are conjugate in $[B_n, B_n]$.
(ii) ([22], Proposition 3.4 and [7], Corollary 2.2) For $n \geq 5$,

$$[B_n, B_n] = [[B_n, B_n], [B_n, B_n]].$$

Therefore, B_n is strongly non-indicable for $n \geq 5$.

Proof. *(i)* Let $\sigma_i \in B_n$ denote the braid which interchanges the i^{th} and the $i+1^{st}$ points for $1 \leq i < n$, and let σ_n be the braid that which interchanges the n^{th} and the first points of the configuration space. Let $\overline{\langle S \rangle}$ denote the normal closure of $\langle S \rangle$. Since the map $A : B_n \to \mathbb{Z}$ sending $\prod_{i=1}^{n} \sigma_i^{m_i}$ to $\sum_{i=1}^{n} m_i$ is the abelianization map, $S = \{\sigma_i \sigma_{i+1}^{-1} : 1 \leq i \leq n-1\}$ is a subset of $[B_n, B_n]$. Consider the following numbers $i, i+1, i+3, j$ modulo n. Since any two σ_i, σ_j of $\sigma_1, \ldots, \sigma_n$ are conjugate, the quotient $B_n / \overline{\langle S \rangle}$ is abelian. Because $\sigma_i \sigma_j^{-1} \in \overline{\langle S \rangle}$ implies $\sigma_i \overline{\langle S \rangle} = \sigma_j \overline{\langle S \rangle}$. So, if $\langle S \rangle$ is a normal subgroup of B_n, then S generates $[B_n, B_n]$.

Claim: The group $\langle S \rangle$ is a normal subgroup of B_n.
Proof of the claim: Note that $n \geq 5$ and $|i - (i+3)|, |(i+1) - (i+3)| \geq 2$, so σ_{i+3} commutes with σ_i, σ_{i+1}.

For all i, j we have $\sigma_i \sigma_j^{-1} \in \langle S \rangle$ because if $i \leq j$, then

$$\sigma_i \sigma_j^{-1} = (\sigma_i \sigma_{i+1}^{-1})(\sigma_{i+1} \sigma_{i+2}^{-1}) \ldots (\sigma_{i+n} \sigma_j^{-1}),$$

and if $i > j$, then $\sigma_i \sigma_j^{-1} = ((\sigma_j \sigma_{j+1}^{-1})(\sigma_{j+1} \sigma_{i+2}^{-1}) \ldots (\sigma_{j+m} \sigma_i^{-1}))^{-1}$.
Moreover, $\sigma_i^{-1} \sigma_{i+1} \in \langle S \rangle$ because

$$\sigma_i^{-1} \sigma_{i+1} = \sigma_{i+3} \sigma_{i+3}^{-1} \sigma_i^{-1} \sigma_{i+1} = \sigma_{i+3} \sigma_i^{-1} \sigma_{i+1} \sigma_{i+3}^{-1} \in \langle S \rangle.$$

So, similar to the reasoning above, any $\sigma_i^{-1} \sigma_j$ is in $\langle S \rangle$.

Then,

$$\begin{aligned}\sigma_j(\sigma_i\sigma_{i+1}^{-1})\sigma_j^{-1} &= (\sigma_j\sigma_{i+3}^{-1})(\sigma_{i+3}\sigma_i\sigma_{i+1}^{-1}\sigma_j^{-1}) \\ &= (\sigma_j\sigma_{i+3}^{-1})(\sigma_i\sigma_{i+1}^{-1})(\sigma_{i+3}\sigma_j^{-1})\end{aligned}$$

and $\sigma_j^{-1}(\sigma_i\sigma_{i+1}^{-1})\sigma_j = (\sigma_j^{-1}\sigma_i)(\sigma_{i+1}^{-1}\sigma_j)$. Thus, $\langle S\rangle \trianglelefteq B_n$ because $\sigma_j\sigma_{i+3}^{-1}$, $\sigma_i\sigma_{i+1}^{-1}$, $\sigma_{i+3}\sigma_j^{-1}$, $\sigma_j^{-1}\sigma_i$,$\sigma_{i+1}^{-1}\sigma_j$ are all in $\langle S\rangle$. This proves the claim.

To complete the proof of part *(i)*, we need to show that all the elements of S are conjugate in $[B_n, B_n]$. By braid relations (see Section 2.1.3.1),

$$(\sigma_i\sigma_{i+1}\sigma_{i+2})\sigma_i\sigma_{i+1}^{-1}(\sigma_i\sigma_{i+1}\sigma_{i+2})^{-1} = \sigma_{i+1}\sigma_{i+2}^{-1}.$$

Note that $(\sigma_i\sigma_{i+1}\sigma_{i+2}\sigma_{i+3}^{-3}) \in [B_n, B_n]$. As σ_{i+3} commutes with σ_i and σ_{i+1},

$$(\sigma_i\sigma_{i+1}\sigma_{i+2}\sigma_{i+3}^{-3})\sigma_i\sigma_{i+1}^{-1}(\sigma_i\sigma_{i+1}\sigma_{i+2}\sigma_{i+3}^{-3})^{-1} = \sigma_{i+1}\sigma_{i+2}^{-1}$$

which gives the elements $\sigma_i\sigma_{i+1}^{-1}$ and $\sigma_{i+1}\sigma_{i+2}^{-1}$ are conjugate in $[B_n, B_n]$.

(ii) Need to write $\sigma_i\sigma_{i+1}^{-1} \in S$ as a commutator in $[B_n, B_n]$. Since $n \geq 5$, there is some j that makes σ_j commute with σ_i, σ_{i+1}. Then, by braid relations (Section 2.1.3.1) again,

$$\begin{aligned}\sigma_i\sigma_{i+1}^{-1} &= (\sigma_i\sigma_{i+1}\sigma_i)(\sigma_i^{-1}\sigma_{i+1}^{-1}\sigma_{i+1}^{-1}) \\ &= (\sigma_{i+1}\sigma_i\sigma_{i+1})(\sigma_i^{-1}\sigma_{i+1}^{-1}\sigma_{i+1}^{-1}) \\ &= [\sigma_{i+1}\sigma_i\sigma_j^{-2}, \sigma_{i+1}\sigma_j^{-1}].\end{aligned}$$

Since $\langle S\rangle = [B_n, B_n]$, we have $[B_n, B_n]$ is perfect. $\square$

Remark 55. *([7], Corollary 2.2) In addition to Proposition 54, any homomorphism from $[B_n, B_n]$ to an abelian group is trivial.*

Lemma 56. *([22], Lemma 3.9) For a group G generated by elements $\tau_1, \ldots, \tau_n$ such that*
(1) τ_i are mutually conjugate for all $i = 1, \ldots, n$
(2) There is $k \geq 2$ with $[\tau_i, \tau_j] = 1$ for $|j - i| \geq k$ (where $|\cdot|$ is the distance in $\mathbb{R}/n\mathbb{Z}$), and for $n \geq 2k + 1$, every homomorphism $f : G \to \mathrm{GL}_2^+(\mathbb{R})$ has abelian image.

Proof. In this proof, we will need Subsection 2.1.4.1. By Remark 18, the preimage of a one-parameter subgroup of the projective linear group $\mathrm{PSL}_2(\mathbb{R})$ in $\mathrm{GL}_2^+(\mathbb{R})$ is abelian. Let π denote the projection from $\mathrm{GL}_2^+(\mathbb{R})$ to $\mathrm{PSL}_2(\mathbb{R})$. If we show that the image of the projection $\overline{f}:=\pi \circ f : G \to \mathrm{PSL}_2(\mathbb{R})$ to $\mathrm{PSL}_2(\mathbb{R})$ is contained in a one-parameter subgroup, then we are done.

Let $\overline{\tau_i}$ denote $\overline{f}(\tau_i)$. By assumption (1) we know any two $\overline{\tau_i}, \overline{\tau_j}$ are conjugate for all $i, j = 1, \ldots, n$ whenever $\overline{\tau_i}, \overline{\tau_j} \neq I$. And if $\overline{\tau_i} = I$ for some $i = 1, \ldots, n$, then all its conjugates, namely all other $\overline{\tau_j}$'s, should be I. Thus, for a non-trivial homomorphism $\overline{f}$, we have $\overline{\tau_i} \neq I$, for all $i = 1, \ldots, n$.

Assume the image of $\overline{f}$ is not contained in a one-parameter subgroup. This means that there are $\overline{\tau_i}, \overline{\tau_j}$ which do not commute. By relabelling if necessary, we may assume that $i = 1$ and $2 \leq j \leq k$ is the minimal index which gives $\overline{\tau_1}, \overline{\tau_j}$ do not commute.

Claim: $j = 2$ for some relabelling.
Proof of the claim: Assume otherwise. Then, either $\overline{\tau_j}$ and $\overline{\tau_{j-1}}$ commute or not.

Case 1: $\overline{\tau_j}$ and $\overline{\tau_{j-1}}$ do not commute. Then, by relabelling again, we get $\overline{\tau_1}, \overline{\tau_2}$ do not commute, i.e., $j = 2$.

Case 2: $\overline{\tau_j}$ and $\overline{\tau_{j-1}}$ commute. Since j was the minimum index giving $\overline{\tau_j}$ does not commute with $\overline{\tau_1}$, $\overline{\tau_{j-1}}$ should commute with $\overline{\tau_1}$. Thus, $\overline{\tau_1}, \overline{\tau_j} \in C_{\mathrm{PSL}_2(\mathbb{R})}(\overline{\tau_{j-1}})$. Note that $C_{\mathrm{PSL}_2(\mathbb{R})}(\overline{\tau_{j-1}}) \subseteq \mathrm{PSL}_2(\mathbb{R})$, and $\mathrm{PSL}_2(\mathbb{R})$ is a one-parameter subgroup, i.e., there is a continuous group homomorphism $\phi : \mathbb{R} \to \mathrm{PSL}_n(\mathbb{R})$, which implies $\overline{\tau_1}, \overline{\tau_j}$ should commute (recall Section 2.1.4.1 here), contradicting the assumption. Hence, $j = 2$ which proves the claim.

By assumption (2) and the fact that $n \geq 2k + 1$, $\overline{\tau_{k+2}}$ commutes with $\overline{\tau_1}$ and $\overline{\tau_2}$. Therefore, $\overline{\tau_1}, \overline{\tau_2} \in C_{\mathrm{PSL}_2(\mathbb{R})}(\overline{\tau_{k+2}})$ which means $\overline{\tau_1}, \overline{\tau_2}$ commute, contradicting the assumption that the image of $\overline{f}$ is not contained in a one-parameter subgroup. $\square$

Plugging $k = 2$ and assigning $\tau_i = \sigma_i$ (where σ_i is as described above) in this lemma, we get the following.

Lemma 57. *([22], Lemma 3.8) For $n \geq 5$, every homomorphism $f : B_n \to \mathrm{GL}_2^+(\mathbb{R})$ has abelian image.*

Theorem 58. *([22], Thurston Stability Theorem) Let M be a manifold and let $x \in M$ be given. For a diffeomorphism g of M fixing x, denote the derivative by $(Dg)_x \in$*

$\mathrm{GL}(T_xM)$. *Then, the group* $\mathcal{G} = \{g \in \mathrm{Diff}^+(M) : g(x) = x, (Dg)_x = I\}$ *is locally indicable (and so are any subgroup of* $\mathcal{G}$*).*

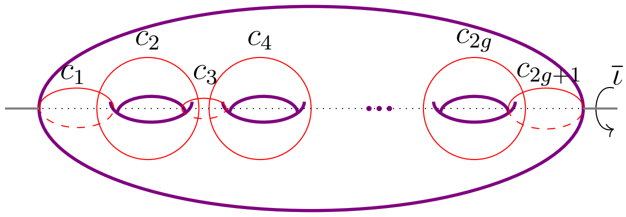

Figure 3.10: Figure of Lemma 61

Lemma 59. *([22], Lemma 4.1) Suppose that there is a realization*

$$\begin{tikzcd} & \mathrm{Diff}^+(\Sigma_g) \arrow[d, two heads] \\ \mathrm{Mod}(\Sigma_g) \arrow[ur, dashed, "\sigma"] \arrow[r] & \mathrm{Mod}(\Sigma_g). \end{tikzcd}$$

Then, the number of fixed points $|\mathrm{Fix}(\sigma(\iota))| = 2 + 2g$ *where* ι *denotes the class of the hyperelliptic involution (see Figure 3.10 where* $\bar{\iota}$ *is a diffeomorphism in the class* ι*).*

Proof. Suppose that the diffeomorphism $\sigma(\iota)$ has n fixed points, say $x_1, x_2, \ldots, x_n$. Consider the induced branched cover $\Sigma_g \to S^2$. Removing these fixed points, we get a $2-$fold covering $\Sigma_g \setminus \{x_1, x_2, \ldots, x_n\}$ of $S^2 \setminus \{\sigma(\iota)(x_1), \sigma(\iota)(x_2), \ldots, \sigma(\iota)(x_n)\}$. Thus we have the equality

$$\chi(\Sigma_g \setminus \{x_1, x_2, \ldots, x_n\}) = 2\chi(S^2 \setminus \{\sigma(\iota)(x_1), \sigma(\iota)(x_2), \ldots, \sigma(\iota)(x_n)\})$$

between the Euler characteristics. So, $2 - 2g - n = 2(2 - n)$, and hence, the number of fixed points of $\sigma(\iota)$ is $n = 2 + 2g$.

Another way: Let h be a Riemannian metric on Σ_g. Since ι is hyperelliptic involution, it is of finite order, so we can consider the metric $h + (\sigma(\iota))^*h + ((\sigma(\iota))^2)^*h + \ldots + ((\sigma(\iota))^{n-1})^*h$ which is $\sigma(\iota)-$invariant where $n = o(\sigma(\iota))$. Therefore, we can consider $\sigma(\iota)$ as an isometry on Σ_g in this metric. Let $x \in \Sigma_g$ be any fixed point of $\sigma(\iota)$. Since $\sigma(\iota)$ is an isometry, it is determined by its derivative at x. This derivative is a 2×2 orthogonal matrix whose determinant is 1 by $\sigma(\iota)$ being orientation preserving.

By $\sigma(\iota)$ being non-trivial, its derivative at x is non-trivial and so x is an isolated fixed point of index 1. By their indices being equal to 1, if we can find the sum of the indices of the fixed points, we get the number of fixed points. By x being arbitrary, all the fixed points of $\sigma(\iota)$ are isolated of the same index. By Lefschetz-Hopf Fixed Point Theorem 35, sum of the indices of fixed points is the same as the Lefschetz number which is:

$$L(\sigma(\iota)) = \sum_{i=0}^{2}(-1)^i Tr((\sigma(\iota))_* : H_i(\Sigma_g, \mathbb{Q}) \to H_i(\Sigma_g, \mathbb{Q})) = 2 + 2g.$$

So, $|\mathrm{Fix}(\sigma(\iota))| = 2 + 2g$. □

Remark 60. *([22], Corollary 4.2) Let $x \in C(\sigma(\iota))$ be an arbitrary element, and let $y \in \mathrm{Fix}(\sigma(\iota))$. Then, $(\sigma(\iota)) \circ x(y) = x(\sigma(\iota))(y) = x(y)$, i.e., $x(y) \in \mathrm{Fix}(\sigma(\iota))$. Since $|\mathrm{Fix}(\sigma(\iota))| = 2g + 2$, we can consider the symmetric group on $\mathrm{Fix}(\sigma(\iota))$ as S_{2g+2}, and get a permutation representation $\rho : C(\sigma(\iota)) \to S_{2g+2}$ of $C(\sigma(\iota))$ on $\mathrm{Fix}(\sigma(\iota))$. In other words, the elements of $C(\sigma(\iota))$ can be represented as permutations on $\mathrm{Fix}(\sigma(\iota))$.*

Let $\alpha, \alpha' \in C(\sigma(\iota))$ be isotopic, then $\rho(\alpha) = \rho(\alpha')$. For a proof of this, see the proof of Lemma 4.5 of [22].

Lemma 61. *([22], Lemma 4.3) For the genus g of Σ_g is greater than or equal to 2, there is a non-indicable subgroup $B \leq C(\iota)$ which is isomorphic to a quotient of B_{2g+2} (here ι denotes the class of the hyperelliptic involution $\bar{\iota}$ in Figure 3.10 is a diffeomorphism in the class ι).*

Proof. Let c_i be simple closed curves as in Figure 3.10. Then, c_i and c_{i+1} intersect only at one point transversely for all $i \in \{1, 2, \dots, 2g\}$, and c_i and c_j does not intersect for $|i - j| \geq 2$. Therefore, the Dehn twists T_{c_i} satisfy the braid relations. Hence, the group $B = \langle T_{c_i} : i \in \{1, 2, \dots, 2g+1\}\rangle \subseteq \mathrm{Mod}(\Sigma_g)$ is a non-trivial quotient of B_{2g+2}. Note that $g \geq 2$ implies $2g + 2 \geq 6$ and so B_{2g+2} is non-indicable (by Proposition 54). By being a quotient of a non-indicable group, B is non-indicable (by Remark 4). ι fixes each c_i, so $T_{c_i} \in C(\iota)$ for each i, and $B \leq C(\iota)$. □

Remark 62. *Consider B as a quotient B_{2g+2}/N. Then, let B' be the restriction of the quotient B to B_{2g+1}, i.e., let $B' := B_{2g+1}/(N \cap B_{2n+1})$. By being a non-trivial quotient* *of a strongly non-indicable subgroup, B' is also strongly non-indicable for $g \geq 2$. By Remark 16, B_{2g+1} fixes a point in B_{2g+2}. Being the quotients of these groups, B' fixes a point in $B \leq C(\iota)$. Considering the images of B and B' under σ, $\sigma(B')$ fixes a point in $\sigma(B) \leq \sigma(C(\iota)) \leq C(\sigma(\iota))$. Therefore, the action of B' on $\mathrm{Fix}(\sigma(\iota))$ has a fixed point.*

Lemma 63. *([22], Lemma 4.5) Let $\mu : B_{2g+2} \to S_{2g+2}$ be the homomorphism which sends the element changing the places of i^{th} and $i+1^{st}$ points in the configuration space $\sigma_i \in B_{2g+2}$ to the permutation $(i\, i+1) \in S_{2g+2}$. Let $\theta : B_{2g+2} \to B = B_{2g+2}/N$ be the map sending σ_i to the Dehn twist T_{c_i} about the curve c_i in the group B (see Figure 3.10 for the curve c_i). Since σ_j generate B_{2g+2} and T_{c_j} generate the group B (see the proof of Lemma 61), the map θ is a homomorphism from B_{2g+2} to B.*
Then, the diagram

$$\begin{tikzcd} B_{2g+2} \arrow[r, "\mu"] \arrow[d, "\theta"'] & S_{2g+2} \\ B \arrow[ur, "\rho \circ \sigma"'] & \end{tikzcd}$$

commutes.

Proof. Note that any two realizations of T_{c_i} are isotopic. By Remark 60, their image should be the same. This means that finding the image of a realization of T_{c_i} under ρ gives the image of any realization of T_{c_i}. Take a realization $\widetilde{T}_{c_i}$ of T_{c_i} which is invariant under the diffeomorphism $\sigma(\iota)$ and whose support is a neighborhood of c_i. Then, the image of $\widetilde{T}_{c_i}$ under ρ is $(i\, i+1)$ which is the same as the image of σ_i under μ, in other words, the diagram commutes. $\square$

The following is a version of Proposition 46 for a compact surface Σ for a different number n of marked points.

Theorem 64. *([22], Theorem 1.1) Let Σ be a compact surface, and let $\mathbf{z}$ be the set of marked points of Σ with $|\mathbf{z}| = n$. Recall the map F in the Section 3.1.1.*
(a) If the boundary $\partial\Sigma = \emptyset$, then $F : B_n(\Sigma) \to \mathrm{Mod}(\Sigma, \mathbf{z})$ is not realized by C^1

diffeomorphisms for all $n \geq 6$.
(b) If the boundary $\partial\Sigma \neq \emptyset$, then $F : B_n(\Sigma) \to \mathrm{Mod}(\Sigma, \mathbf{z})$ is not realized by C^1 diffeomorphisms for all $n \geq 5$.

Proof. *(a) Assume that $\partial\Sigma = \emptyset$ and that for $n \geq 6$, there is a lift*

$$\begin{tikzcd} & \mathrm{Diff}^+(\Sigma, \partial\Sigma, \mathbf{z}) \arrow[d, two heads] \\ B_n(\Sigma) \arrow[ur, dashed, "\sigma"] \arrow[r, "F"] & \mathrm{Mod}(\Sigma) \end{tikzcd}$$

where $\mathrm{Diff}^+(\Sigma, \partial\Sigma, \mathbf{z})$ denotes the group of C^1 diffeomorphisms f from Σ to Σ preserving $\mathbf{z}$ setwise such that the restriction of f to $\partial\Sigma$ is the identity.

Then, by Theorem 53 *(b)*, there is a non-trivial homomorphism

$$\psi : B_{n-1} \to \mathrm{Mod}(\Sigma, \mathbf{z}).$$

For any subgroup H of B_{n-1} witnessing the non-indicablity of B_{n-1} (in particular, by Proposition 54 (ii), H can be taken as $[B_{n-1}, B_{n-1}]$), $H \cap \ker(\psi) \neq H$, so by Remark 4, $B_{n-1}/\ker(\psi)$ is non-indicable. Therefore, the image $\psi(B_{n-1}) \cong B_{n-1}/\ker(\psi)$ of B_{n-1} for $n \geq 6$ is also non-indicable. Thus, $\mathrm{Mod}(\Sigma, \mathbf{z}) \supseteq \psi(B_{n-1})$ is non-indicable.

As B_{n-1} stabilizes $\mathbf{z} \backslash \mathbf{z}_{n-1}$ (by Remark 16 (i)), $\sigma(B_{n-1})$ fixes some point $x \in \mathbf{z} \backslash \mathbf{z}_{n-1}$. Let $D : B_{n-1} \to \mathrm{GL}_2^+(\mathbb{R})$ be the derivative at x. Since for $n \geq 5$, every homomorphism $f : B_n \to \mathrm{GL}_2^+(\mathbb{R})$ should have abelian image (by Lemma 57), the perfect subgroup $[B_{n-1}, B_{n-1}]$ should be in the kernel of D. Then, by Thurston Stability Theorem 58, $[B_{n-1}, B_{n-1}]$ should be locally indicable which is not (by Proposition 54).

(b) $\partial\Sigma \neq \emptyset$, i.e., $n \geq 5$ In this case, $\mathrm{Mod}(\Sigma, \mathbf{z})$ is still strongly non-indicable, and by Remark 16 *(ii)*, the lift $\sigma(B_n)$ fixes some point $x \in \mathbf{z}_{n+1} \setminus \mathbf{z}$. Let $D : B_n \to \mathrm{GL}_2^+(\mathbb{R})$ denote the derivative at x. By Lemma 57, $[B_n, B_n] \subseteq \ker(D)$, and by Thurston Stability Theorem 58 $[B_n, B_n]$ is locally indicable which is not the case. $\square$

3.2.2 Morita's Non-lifting Theorem

Here, we prove Part *(b)* of Theorem 52 in a different way for an arbitrary number of marked points of Σ_g.

Theorem 65. *([22], Theorem 1.2) Let Σ_g be a closed surface of genus g. For $g \geq 2$, the group $\mathrm{Mod}(\Sigma_g)$ does not lift to $\mathrm{Diff}^+(\Sigma_g)$.*

Proof. Recall that for a realization $\sigma : \mathrm{Mod}(\Sigma_g) \to \mathrm{Diff}^+(\Sigma_g)$ has a stongly non-indicable subgroup $\sigma(B') \leq \mathrm{Diff}^+(\Sigma_g)$ where B' is the image of the abovementioned non-indicable subgroup B in B_{2g+1}, i.e., it is a quotient of B_{2g+1} (see Remark 62). So, B' is non-indicable by Remark 4, and $\sigma(B')$ is the image of B' under σ. Moreover, $\sigma(B')$ acts on Σ_g with a global fixed point $p \in \Sigma_g$ (see Remark 62).

Since $\sigma(p) = p$, we can consider the derivative $D_p : \sigma(B') \to \mathrm{GL}_2^+(\mathbb{R})$ at p. We know that the image of B_{2g+1} under $D_p \circ \sigma : B_{2g+1} \to \mathrm{GL}_2^+(\mathbb{R})$ is abelian by Lemma 57. By being a quotient of this group, B' should have abelian image under this map.

Let $P \leq B'$ be a subgroup of B' witnessing the non-indicability of B' (i.e., let P be a non-trivial finitely generated perfect subgroup of B'). Then, the image $\sigma(P)$ of subgroup P under σ should have trivial image under D_p by above. But, by Thurston Stability Theorem 58, $\sigma(P)$ should be locally indicable which is a contradiction. So, there cannot be such a realization σ. $\square$

CHAPTER 4

CONCLUSIONS

In this thesis, we dealt with the answer to the question if a finite index subgroup of $\mathrm{Mod}(\Sigma_g)$ lifts to $\mathrm{Diff}^+(\Sigma_g)$ where Σ_g is a surface of genus g. Morita answered this question for $g \geq 18$ and later for $g \geq 5$ [16]. We considered two different proofs of this result.

In the first proof due to [21], we take a finite index subgroup Γ of $\pi_1(\Sigma_g, z)$ for $z \in \Sigma_g$ and $g \geq 2$, and assume that Γ lifts to $\mathcal{G}(\Sigma_g, z)$. Then, we get a closed annulus $\mathcal{A}$ from $\mathbb{H}^2 \setminus \{z\}$. Then, by Milnor 43 and Milnor-Wood 42 Inequalities, we get the contradiction that the Euler numbers of two boundary components of this annulus cannot be the same. From the result that Γ does not lift to $\mathcal{G}(\Sigma_g, z)$, we get our main result.

For the second proof due to [22], we first show the fact that the braid group B_n is non-indicable. Then, we take a subgroup P of B' witnessing the non-indicability of B' where B' is a quotient of B_{2g+1}. Then, we show that the image of P under a realization σ should be trivial. However, Thurston Stability Theorem 58 implies that $\sigma(P)$ is locally indicable which is a contradiction.

In conclusion, we get $\mathrm{Mod}(\Sigma_g)$ is not realized by orientation preserving diffeomorphisms of Σ_g for $g \geq 2$. Moreover, any finite index subgroup Γ of $\mathrm{Mod}(\Sigma_g, \mathbf{z})$ does not lift to $\mathrm{Diff}^+(\Sigma_g, \mathbf{z})$ where $|\mathbf{z}| \geq 1$ for the same g, and if we take $\mathbf{z}$ of an arbitrary cardinality, then Γ does not lift to $\mathrm{Diff}^+(\Sigma_g, \mathbf{z})$ for $g \geq 6$.

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